THE TIME EVOLUTION OF THE COSMIC MICROWAVE BACKGROUND PHOTOSPHERE

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To Mom and Dad
This paper represents my own work in accordance with University regulations.

Stuart R. Lange
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My explanation of the background material (chapters 1-4) relies heavily on Scott Dodelson’s *Modern Cosmology* (2003). Although I have cited Dodelson’s work numerous times throughout this paper, I here acknowledge its influence on the overall organization and structure of this document and on the background derivations contained therein.

This project required an extensive amount of programming, much of which was built upon pre-existing code. The central implementation in this paper is an extension of the existing CAMB (Code for Anisotropies in the Microwave Background) framework,¹ by Antony Lewis and Anthony Challinor (2000). CAMB, in turn, is based on CMBFAST,² by Uros Seljak and Matias Zaldarriaga (1996). In addition, some of the results in this paper have been derived using the HEALPix³ (Górski et al., 2005) package and the CFITSIO⁴ (Hanisch et al., 2001) library.

Finally, I would like to thank my many friends and family members who have supported me on this project and throughout my time at Princeton. Thank you for making my experience over the last four years so wonderful. Special thanks are due to the Princeton University Band, my true home away from home, for making it all worthwhile. Three cheers for Old Nassau!

¹http://camb.info/
²http://cfa-www.harvard.edu/~mzaldar/CMBFAST/cmbfast.html
⁴http://heasarc.nasa.gov/docs/software/fitsio/fitsio.html
The rapid advance in precision measurements of the cosmic microwave background (CMB), which is exemplified by the recent WMAP results (Spergel et al., 2006), gives us unprecedented ability to consider novel questions about the nature of the CMB. This thesis presents a method to model the evolution of the CMB in time from the point of view of a stationary observer. This model requires extensions both to the existing theory of CMB modeling and the existing implementation. The model has the ability to create representative visualizations of the evolution of the CMB over several time steps. Using the model, we are able to determine that the CMB power spectrum shifts to smaller scales as time progresses, that the late-time ISW effect causes the low-$\ell$ tail to eventually overtake the first acoustic peak in a $\Lambda$CDM universe, and that the parameters required of an experiment designed to measure the time evolution effect cannot be attained in the near future. In addition to presenting the model and the theory behind it, this thesis reviews CMB cosmology from basic principles through the derivation of the line of sight integration approach for CMB modeling.
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Recent experiments researching the cosmic microwave background radiation (CMB), such as the Wilkinson Microwave Anisotropy Probe (WMAP),\(^1\) have increased our understanding of the universe enormously. By studying this faint afterglow of the big bang, these experiments are able to determine important characteristics of the universe — such as its age, expansion rate, density, and composition — to high precision. This rapid increase in knowledge allows us to consider more novel theoretical questions regarding the CMB itself.

In this thesis, I propose to tackle such a question — How does the visible CMB evolve in time from the point of view of a stationary observer? A complete answer to this question requires an understanding of the equations governing the evolution of the components of the universe. This chapter aims to begin the development of that understanding by introducing the basic principles of cosmological physics.

1.1 The Expanding Universe

Concepts such as “The Big Bang” and “The Expanding Universe” are part of common parlance today, but the scale and origins of the universe were hotly debated in the early and middle part of the twentieth century. In fact, the 1920 discussion between astronomers Harlow Shapley and Heber Curtis concerning the nature and scale of the Milky Way galaxy became known as the Great Debate.\(^2\)

The subject of the debate was whether the object Andromeda and other similar “spiral nebulae”, as galaxies were then known, are part of our Milky Way galaxy or rather individual galaxies in their own right, separated from our own by incredible distances. Shapley argued the former point, adding that our Milky Way galaxy itself is of quite large scale, and that our solar system is not located near its center. Curtis countered, placing the solar system near the center of a relatively small Milky Way galaxy that was one of many galaxies in the larger universe. With the benefit of hindsight, we know that Shapley was right about the scale of the Milky Way and our location within it, while Curtis was right about the nature of the “spiral nebulae”.

The Great Debate was not immediately resolved, but the case for a universe consisting of multiple separate galaxies was greatly furthered by Edwin Hubble’s work of the mid-1920s, in which he measured the distances to several nearby “spiral nebulae”, including Andromeda, by observing Cepheid variable stars within the nebulae.\(^3\) His early measurements of Andromeda itself (Hubble, 1929b) revealed it to be much more distant than the maximum extent of the Milky Way galaxy proposed by Shapley, proving that the “spiral nebulae” are in fact distinct galaxies.

\(^1\)http://lambda.gsfc.nasa.gov/product/map/current/
\(^2\)The proceedings of the debate were published a year later (Shapley and Curtis, 1921). For a historical description of the event, see Trimble (1995).
\(^3\)A Cepheid variable star is a standard candle — an object that shines with a known luminosity. Combining this known luminosity with the observed luminosity of the object, the inverse square law gives the distance.
The earlier work of Vesto Slipher (1915) documented the relative velocities of many nearby galaxies, via measurements of their spectral shift. Slipher found that eleven of the fifteen nebulae he studied had redshifted spectra, implying that they are moving away from us.\footnote{Interestingly, the most notable "spiral nebula", Andromeda, was found to have a blueshifted spectrum, implying a velocity of approach. This is due to the gravitational attraction between our galaxy and the relatively nearby Andromeda. We will discuss redshift further in §1.2.2.} Hubble combined this work with his own distance measurements (Hubble, 1929a), revealing a simple linear relationship:

\[ v = H_0 r \]  

where \( v \) is the recessional velocity of galaxy, and \( r \) is its distance. The constant \( H_0 \), which became known as Hubble's constant, is traditionally measured in kilometers per second per megaparsec. Hubble's Law, as equation 1.1 is known, states that the farther away a galaxy is from us, the faster it moves away from us.

This effect can be explained by the "expanding universe" model, in which the fabric of space itself stretches as time progresses. The effect is illustrated in figure 1.1, which shows two objects, A and B, which are at rest with respect to the comoving grid (the "fabric" of space, if you will). As time progresses, space itself expands, increasing the physical distance between A and B.\footnote{The figure and discussion are adapted from Dodelson (2003)}

\[ r(t) = a(t) x(t) \]  

Let the comoving distance (as measured on the comoving grid) between two objects be \( x \), and the physical distance between them be \( r \). The two are related by the dimensionless scale factor, \( a \):

\[ r(t) = a(t) x(t) \]  

Note that all three elements of this equation can change with time — the expanding universe is parameterized by a scale factor that monotonically increases with time. In the context of the expanding universe, we can reinterpret the Hubble law as a special case of equation 1.2. Differentiating
equation 1.2 with respect to time, we get:

$$\frac{dr}{dt} = a(t) \frac{dx}{dt} + \frac{da}{dt} x(t)$$

(1.3)

If we consider only observers that are at rest with respect to the comoving grid, $dx/dt = 0$. Also, note that $dr/dt = v(t) = H(t)r(t) = H(t)a(t)x(t)$ (where we allow the Hubble “constant” to change with time). Hence:

$$H(t) = \frac{1}{a(t)} \frac{da}{dt}$$

(1.4)

This form makes it clear that the Hubble constant $H(t)$ in fact parameterizes the rate of expansion of the universe (note that, fundamentally, $H(t)$ has units of inverse time). Therefore, we normally refer to $H(t)$ as the Hubble rate. The inverse of the Hubble rate gives the characteristic time scale over which the universe expands appreciably. The Hubble rate is often expressed as a dimensionless parameter $h$, which relates to the Hubble rate of equation 1.4 as follows:

$$H(t) = h(t) \times 100 \frac{\text{km/s}}{\text{Mpc}}$$

(1.5)

1.2 A Notational Interlude

Before moving on, we need to pause to establish some notational and physical underpinnings to inform the rest of our discussion. In equation 1.1, I snuck in the notational convention that the subscript “0” denotes the present time. Hence, $t_0$ denotes the present time, and $H(t_0) = H_0$ is the Hubble rate today. The dimensionless scale factor, $a(t)$, can be arbitrarily assigned; the convention is to set it to unity at the present time: $a(t_0) = a_0 = 1$.

As is common practice in cosmology, from this point on, I set the speed of light, $c$, equal to one. Making the speed of light dimensionless forces us to measure time in units of distance.\footnote{Equivalently, we could measure distance in units of time — using the familiar “light year” for example. However, common practice in cosmology is to think of time as a distance.} For most of our discussions, explicit units will not be necessary, but it is useful to think of both distance and time being measured in megaparsecs.

1.2.1 Conformal Time

Making use of our new unit system, we note that in a small amount of time $dt$, light travels a physical distance $dr = dt$. Hence by equation 1.2, the light travels a comoving distance $dx = dt/a(t)$. Integrating this comoving light travel distance since the beginning of the universe gives

\footnote{Alternatively, $t_0$ gives the age of the universe.}
us the conformal time. Adopting \( \tau \) as the symbol for conformal time, we have:

\[
\tau(t) = \int_0^t \frac{1}{a(t')} dt'
\]

(1.6)

\( \tau \) will be useful as a time variable in our equations. Also, at conformal time \( \tau \), the maximum comoving distance light could have traveled since the big bang is \( \tau \), and hence, information could never have been exchanged between objects separated by a greater comoving distance. We can think of \( \tau \) as the comoving horizon – objects separated by a comoving distance less than \( \tau \) are in causal contact, while objects separated by a comoving distance greater than \( \tau \) are not. The comoving horizon today is given by \( \tau_0 = \tau(t_0) \), the conformal time since the big bang.

### 1.2.2 Redshift

Another useful measure of “time” is the redshift \( z \) to a particular epoch. The use of redshift has deep roots in observational astronomy — redshift is the dimensionless measure of the amount by which an object’s spectrum is Doppler shifted due to its velocity relative to the observer. If an object emits radiation at wavelength \( \lambda_e \), but we observe it at wavelength \( \lambda_0 \), the redshift is given by:

\[
z = \frac{\lambda_0 - \lambda_e}{\lambda_e}
\]

(1.7)

This means that objects that are stationary with respect to the observer have redshift zero. However, an object that is stationary with respect to the observer in comoving coordinates will still be subject to Doppler shift due to the expansion of the universe. This redshift that is purely due to the expansion of the universe is called the cosmological redshift, and is a useful measure of distance and time that is closely related to the scale factor.

To begin exploring the cosmological redshift, consider an object at a fixed comoving distance \( x \) from the observer, and consider two successive wave crests emitted by the object. Let the wavelength at emission be \( \lambda_e \) and the observed wavelength be \( \lambda_0 \). The first crest is emitted at physical time \( t_e \), and observed at physical time \( t_0 \). The second is emitted at time \( t_e + \lambda_e \) and observed at time \( t_0 + \lambda_0 \) (remember that \( c = 1 \)). Since both wave crests travel the same comoving distance, they take the same amount of conformal time to make the trip:

\[
\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e + \lambda_e}^{t_0 + \lambda_0} \frac{dt}{a(t)}
\]

(1.8)

\[
\int_{t_e}^{t_e + \lambda_e} \frac{dt}{a(t)} = \int_{t_0}^{t_0 + \lambda_0} \frac{dt}{a(t)}
\]

(1.9)

where in the second equality we have subtracted off the conformal time during which both crests are “in flight”, demonstrating that the conformal period of the radiation is the same at the emitter and the observer. We now note that the scale factor, \( a(t) \), will not change significantly over the

---

8There is no standard choice of symbol for conformal time. I choose \( \tau \) in keeping with the Seljak and Zaldarriaga (1996), but others use different symbols – for example, Dodelson (2003) uses \( \eta \) for conformal time, reserving \( \tau \) for optical depth.

9This explanation is adapted from Ryden (2003), section 3.4.
course of either integral in the above expression. Hence we allow \( a(t) \) to be constant in each integral:

\[
\frac{1}{a(t)} \int_{t}^{t+\lambda} dt = \frac{1}{a(t_0)} \int_{t_0}^{t_0+\lambda_0} dt
\]

(1.10)

But now the integrals simply evaluate to the physical periods (wavelengths) of the radiation at emission and observation, respectively, and hence:

\[
\frac{\lambda_0}{\lambda_0} = \frac{1}{a(t)}
\]

(1.11)

And now, recalling equation 1.7, and generalizing the time of emission, we have:

\[
z(t) + 1 = \frac{1}{a(t)}
\]

(1.12)

Hence, high redshifts correspond to small scale factors and early times.

### 1.3 Implications of Expansion

One of the many intriguing features of the expanding universe theory is that it requires that the expansion had a beginning. For, if we rewind the history of our monotonically expanding universe, there must have been a time when the entire “fabric” of space was crammed into an incredibly small physical volume at extreme densities and temperatures. Rewinding the clock back even further requires that the universe emerged from a singularity. This theory was first expressed by Georges Lemaître (1927), who actually derived the expanding universe from Einstein’s equations of general relativity (as we will do in chapter 2). The term “big bang” was first used to describe this fiery beginning as a sarcastic jab by an opponent of the theory, British astronomer Fred Hoyle, on a BBC television program in 1949. Although it was originally meant as an insult, the term “big bang theory” has stuck as a moniker for the theory of the expanding universe that emerges from a singularity.

Hoyle did not disagree with Hubble’s observations and Lemaître’s calculations that indicated the expansion of the universe — rather, he proposed that mass was constantly being added to the universe as it expanded, so that although galaxies continued to move apart, new galaxies would continually form in the intervening space, keeping the universe in the same steady state forever. The theory was first proposed by Hoyle’s colleagues Thomas Gold and Hermann Bondi (1948). This “steady state theory” allows for the universe to be infinitely old, and hence does not require that the universe had a beginning.

The largest piece of supporting evidence for the big bang theory was the discovery of the cosmic

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\(^{10}\)This is justified because the period of a photon is on the order of \(10^{-15}\) seconds while the time that it takes the universe to expand appreciably, the Hubble time \((1/H(t))\), is on the order of \(10^{10}\) years.

\(^{11}\)Not all theories require a singular beginning. In cyclic theories, for example, the density and temperature are never infinite. Rather, the universe expands and contracts in a never ending cycle. We will discuss this theory further in §1.4.

\(^{12}\)Lemaître also published an English translation (1931) in which the term “primeval atom” was used to describe the singularity.
microwave background by Arno Penzias and Robert Wilson of Bell Labs, who discovered it quite by accident as unaccounted-for 3 Kelvin noise in their microwave antenna (Penzias and Wilson, 1965). At precisely the same time, Princeton physicists were developing the theoretical basis for the existence of the CMB (Dicke et al., 1965) and building their own antenna for measurement purposes.

Big bang cosmology predicts the existence of this background radiation. In the extremely hot, dense, and energetic environment of the universe immediately following the big bang, the universe was simply a sea of simple energetic particles, such as photons, protons, and electrons. In standard cosmological nomenclature, protons and electrons (and all luminous matter) are lumped into the category “baryons,” so this early sea of high-energy particles is known as the photon-baryon plasma. In this state, photons constantly scatter off the free electrons, so the mean free path of a photon is fairly short, and the photons and electrons remain in thermal equilibrium. As the universe continues to expand, it cools, and eventually the temperature becomes low enough that protons can capture electrons, forming Hydrogen atoms. After this event, called recombination, which occurred roughly 400,000 years after the big bang (at redshift $z_{\text{rec}} \approx 1100$), photons only scatter if they have one of the specific energies required to cause an energy level transition in Hydrogen. Other photons simply pass through the Hydrogen atom unabated. This implies that the universe becomes transparent at recombination, after which the the relic photons from the big bang simply free stream through the universe along straight paths. As the universe continues to expand, these once highly energetic photons decrease in energy, forming the cosmic microwave background that we can observe today.

At the time of recombination, we can imagine every point in space as a point source for CMB photons that emanate in all directions. Complementarily, after recombination, we can think of every point in space as a sink for CMB photons that all “last scattered” some fixed distance away (determined, to first order, by the time since recombination). Hence, every point in spacetime (including our own), has with it an associated surface of last scattering from which the observable CMB emanated. The surface of last scattering defines the extent of the visible universe for each point in spacetime, since the CMB is the oldest electromagnetic radiation that can be observed at any given point. These concepts are illustrated in figure 1.2. When experiments such as WMAP measure the CMB, they “see” this surface of last scattering.

Since the photon-baryon plasma was in thermal equilibrium before recombination, the big bang theory predicts that the observed CMB will exhibit a blackbody radiation spectrum. The Cosmic Background Explorer (COBE) experiment of the 1990s dramatically confirmed this prediction (Mather et al., 1994) — as seen in figure 1.3. COBE confirmed that the CMB temperature is nearly isotropic — 2.725 Kelvin in all directions (Mather et al., 1999). COBE’s Differential Microwave Ra-

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13 At very early times, densities are so high and space is so compact that General Relativity and Quantum Mechanics come into conflict and cannot predict what happens. Hence, by “immediately following”, I mean that point in time at which the universe has cooled down enough that our existing theories are valid. The time at which this occurs is about $10^{-32}$ seconds after the big bang in the inflationary model, and about $10^{-25}$ seconds after the big bang in the cyclic model. We discuss both theories in §1.4.

14 This nomenclature conflicts with particle physics, where “baryon” is the term used for 3-quark particles, which electrons decidedly are not. However, we will always use “baryons” in the cosmological sense in this paper.

15 Again, the nomenclature is a bit strange, since this was the first time in the history of the universe that protons and electrons combined.

16 Their wavelengths are increased by the expansion of the universe and their density drops due to expansion.
Figure 1.2: CMB sources and sinks. Figure (a) shows a point in space at the time of recombination, with photons free streaming away through the now-transparent universe. Figure 1.2b shows an observer at some later time observing CMB photons that all originated from the surface of last scattering surrounding him.

Figure 1.3: A fit of the COBE (Cosmic Background Explorer) FIRAS (Far Infrared Absolute Spectrophotometer) CMB spectrum measurements to a blackbody at 2.7K. The experimental data points and error bars are obscured by the theoretical curve (Fixsen et al., 1996).
diometer (DMR) experiment discovered extremely small deviations from perfect isotropy whose rms value over the whole sky is $30\mu$K (Smoot et al., 1992) – or about one part in $10^5$.

### 1.4 The Horizon Problem

The stunning isotropy of the CMB is spectacular evidence for the big bang theory of cosmic origins. However, it also raises alarm bells. CMB photons that reach us today from opposite sides of the sky are just coming into contact with us — how could they ever have been in contact with each other to reach thermal equilibrium? Figure 1.4 illustrates this horizon problem raised by the isotropy of the CMB. In the two figures (plotted in comoving coordinates, not to scale), we observe CMB photons today that originated from a last scattering surface surrounding our location. Those CMB photons are traced back to the points at which they last scattered, and the extent of the comoving horizon at that point in spacetime is shown. Note that photons that are separated by small angles, as in figure 1.4a, did in fact have overlapping horizons at recombination, and hence were in causal contact and could have reached thermal equilibrium. However, photons that are separated by large angular scales, as in figure 1.4b, do not have overlapping horizons and therefore were not in causal contact before recombination. We will derive in section §2.5.3 that the maximum angular separation at which photons arriving today could have been in causal contact before recombination is about one degree.

![Figure 1.4: The horizon problem.](attachment:figure1_4.png)

This argument shows that we cannot reasonably expect CMB photons from all over the sky to be at exactly the same temperature; since they were not even in causal contact, there would be no

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17The horizon problem also encompasses the problem raised by the fact that the entire universe (including the CMB, but also including galaxy distributions) looks the same in all directions, and presumably from all points in space.
way for them to reach thermal equilibrium. But yet, as we saw in the previous section, the CMB is a near-perfect isotropic blackbody at 2.725 Kelvin. How can these conflicting observations possibly be reconciled?

Alan Guth provided a solution, dubbed the inflationary universe (or just “inflation” for short), that solves the horizon problem (Guth, 1981). In the inflationary model, the visible universe (that is, those bits of the universe that are visible to us today), was extremely tiny (much more so than in the standard big bang model) until about $10^{-34}$ seconds after the big bang. In fact, it was so small that the entire visible universe was in causal contact, allowing it to come to thermal equilibrium. At roughly $10^{-34}$ seconds, the universe entered into a period of extremely rapid, exponential expansion that lasted for about $10^{-25}$ seconds. It is estimated that the universe expanded by a factor of $10^{50}$ during inflation. After this epoch, the entire visible universe was still at the same temperature (because it had reached thermal equilibrium before inflation). This allows the photon-baryon plasma to remain at a uniform temperature all the way to recombination, accounting for the observed isotropy of the CMB.

Another solution to the horizon problem is provided by Paul Steinhardt and Neil Turok, dubbed the cyclic model (Steinhardt and Turok, 2002). In this model, the universe has no beginning and no end, but rather undergoes a never-ending series of expansion and contraction cycles. The density and temperature in the cyclic universe are always finite — the universe does not emerge from a poorly-defined infinitely dense initial state. Rather, the “bounce” from one cycle to another occurs before the universe contracts to a singularity. The model requires that the universe undergo a long period of accelerated expansion to dilute the “debris” of the current cycle and re-set the universe to a vacuum state in preparation for the next cycle. The next cycle then emerges from a homogenous “initial” state, solving the horizon problem.

Throughout this work, we will typically assume the inflationary model, but we should keep in mind that inflation is far from a proven fact.

### 1.5 Anisotropies

In addition to explaining the large-scale isotropy of the CMB, inflationary theory accounts for the small scale deviations from the mean temperature, or anisotropies, that have been observed. Before the inflationary epoch, the visible universe was so small that quantum fluctuations caused localized deviations from thermal equilibrium. Inflation magnified these quantum fluctuations, setting up initial perturbations to the otherwise isotropic photon-baryon plasma. These perturbations manifest themselves as deviations from the average density (and hence temperature) of the photon-baryon plasma. Before recombination, the photons and baryons remain (locally) in equilibrium, so a region that is over-dense in matter will also be over-dense in photons. The high density of such regions increases their gravitational pull, which tends to draw more matter and photons in. However, the temperature increases along with the density, causing the pressure (mostly supplied by the photons) to rise. This pressure tends to drive matter and photons away from over-

---

18Inflation also solves the flatness problem, which asks why the universe remains so near geometrical flatness (an unstable equilibrium) today, 14 billion years after the big bang. Guth’s original theory was revised to its present form by Albrecht and Steinhardt (1982).

19See Carroll and Ostlie (Carroll and Ostlie, 1996), §28.2.
dense regions and into under-dense regions. These competing forces cause the perturbations in the photon-baryon plasma to oscillate in time.

Whenever we work with perturbations and anisotropies, we will define an anisotropy function that gives the deviation from the mean at a given point in time. For example, the perturbation to the photon temperature as a function of space and time is given by

\[ \Delta T(\vec{x}, t) = T(\vec{x}, t) - \langle T(t) \rangle \]  

(1.13)

where \( \langle T(t) \rangle \) denotes the average photon temperature (over the entire universe) at the time under consideration.\(^{20}\) The vector \( \vec{x} \) denotes comoving position in three-dimensional space.

At this point, it is useful to introduce the concept of decomposition into Fourier modes. Doing so allows us to work with perturbations on a certain scale, rather than at a specific point in space. If we let \( \Delta(\vec{x}, t) \) be the anisotropy function of a certain component of the universe (photons, baryonic matter, etc.) as a function of space and time, the Fourier transform of the perturbation is given by

\[ \Delta(\vec{k}, t) = \frac{1}{(2\pi)^3} \int e^{-i\vec{k} \cdot \vec{x}} \Delta(\vec{x}, t) d^3x \]  

(1.14)

and the inverse transform is given by:

\[ \Delta(\vec{x}, t) = \int e^{i\vec{k} \cdot \vec{x}} \Delta(\vec{k}, t) d^3k \]  

(1.15)

The Fourier wavevector \( \vec{k} \) has units of inverse length.\(^{21}\) Its magnitude, \( |\vec{k}| = k \), is the inverse of the wavelength of the mode. We can think of \( 1/k \) as the distance between two adjacent “hot spots” in the given mode. Hence, small-\( k \) modes correspond to large-scale anisotropies, while high-\( k \) modes correspond to small-scale anisotropies. The direction of the Fourier wavevector, \( \vec{k}/k = \hat{k} \), gives the orientation of the mode. \( \Delta(\vec{k}, t) \) gives the amplitude of the mode.

The oscillations over time in the photon-baryon plasma due to the interplay of gravity and pressure manifest themselves as oscillations in the amplitude of each mode. Oscillations will only affect small-scale (large-\( k \)) modes, those that are smaller than the horizon at recombination. For modes of wavelength larger than this horizon, adjacent “hot spots” are not in causal contact with each other during the photon-baryon plasma epoch, and therefore these modes do not undergo oscillations. For modes that do undergo oscillation, we can think of each Fourier mode as a standing wave — it has a fixed spatial wavelength \( (1/k) \), and an amplitude that oscillates in time (overdense regions become underdense, and vice versa). This is similar to the way sound waves propagate through a musical instrument,\(^{22}\) and it is useful to picture these oscillations as standing sound waves in the photon-baryon plasma. If we let the speed of sound in the plasma be \( v \), then a given \( k \)-mode

\(^{20}\)We will have to revise this definition later to make \( \Delta_T \) a function of the photon momentum, as well, but this simple form will suffice for the discussion in this section.

\(^{21}\)The placement of the required factors of \( 2\pi \) is somewhat arbitrary. Mathematicians prefer to split them over both transformations, giving each a factor of \( (2\pi)^{-3/2} \). However, our calculations will remain cleaner if we consolidate the factor in the “forward” direction. This follows the convention of Seljak and Zaldarriaga (1996), who in turn follow the convention of Ma and Bertschinger (1995). Dodelson (2003) uses the opposite convention, placing the factor of \( (2\pi)^{-3} \) in the “backward” direction of the transform.

\(^{22}\)See Hu and White (2004) for an accessible description of the analogy.
oscillates in time with frequency $\nu k$. Hence, small-$k$ (large scale) modes oscillate slowly and large-$k$ (small scale) modes oscillate quickly.

This time-dependence has interesting implications for the time of recombination. At recombination, when the photons no longer interact strongly with the baryons (since the electrons are bound to Hydrogen atoms), the pressure that prevented overdense regions from becoming increasingly dense suddenly vanishes. Therefore, regions that happened to be overdense at the time of recombination continued to accrue matter unchecked — eventually growing into the large-scale structures of today. The CMB photons, however, simply free stream through the universe after recombination, carrying with them whatever energy they happened to have at that time. Hence, by observing the CMB today, we get a snapshot of the anisotropies in the photon-baryon plasma at the time of recombination.

If we picture the situation in Fourier space, we realize that only certain $k$-modes were maximally expressed at the time of recombination. Figure 1.5 traces the amplitude of several (carefully chosen) $k$-modes from inflation to recombination. In the figure, we scale time such that $t = 1$ at recombination, to simplify the discussion. The mode with $k = 1/4 \nu$ (frequency $1/4$) gets through one quarter of an oscillation and therefore has amplitude zero at recombination. Therefore, perturbations at this scale were not expressed at the time of recombination, so we expect not to see them in the CMB today. The $k = 1/2 \nu$ mode goes through half a full oscillation by recombination (so initially overdense regions have become underdense, and vice versa, for this mode), and hence should be expressed in the CMB today. Going back to the musical analogy, we can think of $k = 1/2 \nu$ as the “fundamental” mode of the oscillations. Modes whose frequencies (wavenumbers) are integral multiples of the fundamental frequency will be expressed, while modes whose frequencies are half-integral multiples of the fundamental will be suppressed. We can see this in the figure — the $k = 3/4 \nu$ mode will be suppressed, while the $k = 1/\nu$ mode will be expressed.

This regularly spaced series of peaks and troughs in the anisotropy spectrum of the CMB is the defining characteristic of acoustic oscillations in the primordial photon-baryon plasma. Since inflationary theory predicts such oscillations, the appearance of such a spectrum in CMB measurements would be strong evidence for the theory. As we shall see in the next section, such a spectrum does indeed appear in modern measurements of the CMB.

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23To be clear, I do not mean to imply that recombination happened instantly. However, it did occur over a very short cosmological time-span, so to first order, we can speak of recombination as a well-defined event.
24This energy is reduced over time due to the expansion of the universe, but this affects all CMB photons equally and therefore does not affect anisotropy. They are also affected by post-recombination processes (such as reionization), but the dominant effect on CMB anisotropy is their energy the time of their release, and the potential of the region from which they were released.
25Note that this discussion of the evolution of anisotropy modes is for a non-physical toy model. It is only meant to illuminate the concept that certain related $k$ modes are highly expressed at recombination, while others are not.
26Again, the model depicted here is non-physical and only designed to help build our intuition about the behavior of different $k$-modes. In reality, modes begin with a slightly non-zero amplitude which grows for some time (due to the influence of gravity), before decreasing due to high pressure, rather than beginning with maximal amplitude that immediately decreases, as is depicted here.
27We should note that the cyclic model also predicts these oscillations.
CHAPTER 1. INTRODUCTION TO CMB COSMOLOGY

Figure 1.5: Representative anisotropy modes in a toy model. Here, time 0 is meant to be immediately after the inflationary epoch, when the initial perturbations have just been set up, and time 1 is recombination. The fundamental mode is the $k = 1/2v$ mode — so we expect modes with $k = n/2v, n \in \mathbb{Z}$ to be expressed in the CMB spectrum.

1.6 CMB MEASUREMENT

When we measure the cosmic microwave background today, the result is a temperature map defined on the surface of last scattering:

$$\delta T(\theta, \phi) \equiv T(\theta, \phi) - \langle T \rangle$$

(1.16)

where $\phi$ is the azimuthal angle, $\theta$ is the polar angle, and as is our custom, we have defined a temperature anisotropy function by subtracting off the average CMB temperature. To compare our CMB measurements to the theory of acoustic oscillations developed in the previous section, we would like to look at CMB anisotropies as a function of their scale on the sky, rather than their position. Since $\delta T$ is defined on a sphere, we cannot use the Fourier transform. However, we can decompose the sky map into spherical harmonics:

$$\delta T(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi)$$

(1.17)

where $Y_{\ell m}$ is the spherical harmonic of degree (or multipole moment) $\ell$ and order $m$. The coefficients $a_{\ell m}$ carry all the original information contained in our sky map $\delta T$, but in a form that allows us to more easily consider anisotropies as a function of their scale. Note that the $a_{\ell m}$ also carry the units.

$^{28}$The spherical harmonics are defined for $\ell, m \in \mathbb{Z}$ with $\ell \geq 0$ and $m \in (-\ell, \ell)$.
of the temperature anisotropy (typically Kelvin or microKelvin). A spherical harmonic of multipole \( \ell > 0 \) oscillates with an angular period of about \( 2\pi / \ell \) radians, or 360/\( \ell \) degrees, and therefore the set of \( a_{\ell m} \)s at a given multipole determines the strength of the CMB oscillations at that angular scale. The angular scale of the observed CMB anisotropy also relates to the wavenumber of the inherent temperature anisotropy. A given \( k \)-mode fluctuates with a comoving wavelength of \( 1/k \), and since the surface of last scattering is at approximately a comoving distance of \( \tau_0 \), we expect a mode with wavenumber \( k \) to manifest itself as an angular anisotropy of angular scale \( 1/k\tau_0 \), or equivalently, at multipole \( \ell \approx 2\pi k\tau_0 \). Hence, the acoustic peak pattern that is present in the inherent photon temperature anisotropies should manifest themselves in the CMB.

To get a handle on the average strength of the CMB anisotropy at a given multipole, we consider the statistical properties of the set of \( a_{\ell m} \)s at that multipole. Since our anisotropy function has zero mean, the distribution of \( a_{\ell m} \)s has zero mean. However, the variance is nonzero, and it is this variance that indicates the “power” of the anisotropies at a given multipole.\(^{29}\) We denote the variance as \( C_\ell \):

\[
C_\ell = \langle a_{\ell m} a_{\ell m}^* \rangle
\]

(1.18)

where the variance carries units of temperature squared. When given as a function of multipole moment, \( C_\ell \) is known as the temperature angular power spectrum (or just the power spectrum in the appropriate context). I have been intentionally ambiguous as to the exact meaning of the average taken in equation 1.18. This is because it has somewhat of a double meaning. The observational interpretation is that we average over all observed \( a_{\ell m} \)s at a particular \( \ell \). The theoretical interpretation is that \( C_\ell \) is in fact the fundamental quantity (it is the quantity that can be predicted by the theory we will develop in the following chapters), and a particular realization is created by sampling \( a_{\ell m} \)s from the distribution determined by \( C_\ell \) — hence the average in equation 1.18 is taken over all possible sky maps in a given universe. While the first meaning is more intuitive, the second meaning is correct. Consider what would happen, for example, if we rotated the coordinate system we use to define the CMB sky map, or if we were at a different location in the universe. Clearly, the specific \( a_{\ell m} \)s measured in each case would be different, but their statistical properties, the \( C_\ell \)s, would be the same. From this standpoint, we see that our ability to measure the fundamental quantity \( C_\ell \) is in fact limited by our ability to sample independent \( a_{\ell m} \)s at a given multipole — for, we only have one realization to sample, so we can only measure \( 2\ell + 1 \) independent \( a_{\ell m} \)s at each \( \ell \).\(^{30}\) This inherent uncertainty in our measurement, which scales like \( \sqrt{2/((2\ell + 1))} \), is known as the cosmic variance.\(^{31}\)

Figure 1.6 is a plot of the power spectrum observed by WMAP, along with the best-fit theoretical model. This is representative of power spectrum plots we will see later on. Multipole is plotted on a logarithmic scale on the horizontal axis, and the anisotropy power, which is given as \( \ell(\ell+1)C_\ell/2\pi \) in \( \mu K^2 \), is plotted on the vertical axis. The series of peaks and troughs that we expect is clearly

\(^{29}\)There is an inherent assumption here that the distribution is Gaussian. This is an assertion that has been upheld by observational evidence (Komatsu et al., 2003).

\(^{30}\)Actually, the \( a_{\ell m} \)s with \( m < 0 \) are not independent of those with \( m > 0 \), so we typically use the value \((2\ell + 1)/2\) as the number of “independent” \( a_{\ell m} \)s.

\(^{31}\)A somewhat unfortunate choice of nomenclature, since the power spectrum is so linked to the statistical notion of variance. Remember that “variance” in this context is meant to mean “uncertainty”.
visible in the experimental and theoretical data.

The many features of the CMB power spectrum make it an extremely powerful tool for determining the cosmological parameters (expansion rate, density, composition, etc.). Each parameter affects the theoretical model that produces the best-fit curve in figure 1.6 in a unique and well-defined way. This feature of the theory and the ability to quickly compute theoretical power spectra, combined with the rapidly increasing precision with which CMB observations can be made, are responsible for the major leap forward in our knowledge of the universe over the past decade.

1.7 THE NEXT STEP

In this section, we have addressed all the basic principles behind the study of the CMB. Our overall goal for this project is to model the behavior of the observed CMB sky map from a fixed (in comoving coordinates) vantage point as time progresses. For such an observer, the comoving distance to the surface of last scattering increases with time (as we have already seen, this distance is approximately the conformal time, $\tau_0$). We can think of the CMB image that we see today as a photosphere that is expanding through the universe, always moving farther and farther away. As the photosphere recedes, we see a more distant slice of the primordial universe.

As we saw in the previous section, CMB temperature perturbations of comoving wavenumber $k$ manifest themselves as anisotropies on the sky at multipole $\ell \approx 2\pi k \tau_0$. Hence, to zeroth order, as time progresses, we expect features of the power spectrum to shift to higher multipoles linearly with conformal time. This effect is illustrated in figure 1.7. In the figure, a representative one-dimensional $k$-mode of the temperature perturbation is shown oscillating from hot (red) to cold (blue) across the universe. As the CMB photosphere expands with time, photons arriving from the same “stripes” of the $k$ mode arrive at a smaller angular separation.

In the following chapters, we will develop a more detailed physical theory of the evolution of the universe and its components. This will allow us to push the model out to future times so that we can obtain a more precise understanding of how the CMB evolves in time.
Figure 1.6: The CMB power spectrum from WMAP three-year data. The points with error bars are the WMAP data, and the curve is the best fit cosmological model. The blue area is the cosmic variance allowed by the best-fit model. Credit: NASA/WMAP Science Team. From the WMAP three year result press release: http://map.gsfc.nasa.gov/m_ig/060911/PowerSpectrum.pdf

Figure 1.7: The recession of the CMB photosphere. In (a), we see two photons arriving from “hot spots” of the pictured $k$-mode at the present time. In (b), we see two photons arriving from the same “hot spots” some time in the future. Since the photosphere has expanded, the photons arrive at a smaller angular scale.
In this chapter, we begin to develop the physical and mathematical tools that will be required to make predictions about the CMB spectrum. Here, we seek to become comfortable with Einstein’s theory of General Relativity\(^\text{1}\) and some of its basic implications. In Einstein’s formulation, gravity is not a “force” acting between two massive objects, as it is in Newtonian theory. Rather, matter and energy act to curve spacetime itself, and objects simply travel along “straight lines”, or geodesics, in this distorted spacetime. To summarize: “Space-time tells matter how to move. Matter tells space-time how to curve” (Misner et al., 1973).

As we explore the theory of General Relativity in this chapter, as an example, we will calculate the evolution of a zero-order expanding universe that is isotropic and homogenous (of uniform energy density). This is more than just an exercise, as this model actually provides a surprisingly good approximation of our universe, which becomes increasingly accurate at early times (consider the homogeneity of the photon-baryon plasma, which we have already encountered). When we study deviations from homogeneity in the next chapter, we will do so by allowing slight perturbations from the zero-order universe we develop here. The ultimate goal of this chapter is to make use of the equations of General Relativity to derive the Friedmann equation, which determines the evolution of the scale factor \(a(t)\) in our homogenous and isotropic universe. We then make use of the Friedmann equation to derive some useful results in this zero-order universe.

### 2.1 Components of the Universe

Before diving into General Relativity, we study some basic properties of the zero-order homogenous universe. Such a universe has an energy density that is constant over space, but varies as a function of time: \(\rho(t)\). It is useful to split this energy density into the densities of the constituent species in the universe, so that we can consider the evolution of each separately. Densities are defined as energy per unit volume in physical coordinates.\(^2\) In our derivations, I will often keep time dependence implicit.

Before we begin to describe specific species, let us consider the formalism for computing the energy density of a generic species — call this energy density \(\rho_i\). The distribution function, \(f_i(\vec{x}, \vec{p})\), of the species gives its number density in terms of the six-dimensional position-momentum phase-space volume. Hence, to find the total number of particles, we would integrate \(f_i\) over all possible positions and momenta. Likewise, to find the energy density, \(\rho_i\), we would integrate \(f_i\) times the energy of one particle (at that location in phase space), over all momenta (note that in general, \(\rho_i\)

\(^1\)First published in (Einstein, 1916).

\(^2\)The density in terms of comoving volume is constant over time for all species.
can vary in space; we will simplify to the homogenous universe case later):

\[ \rho_i(\vec{x}) = g_i \int \frac{d^3p}{(2\pi)^3} f_i(\vec{x}, \vec{p}) E(p) \]  

(2.1)

where \( E(p) = \sqrt{m_i^2 + p^2} \) is the energy of an individual particle as a function of its momentum (note that it is not dependent on the direction or position), and \( g_i \) gives the quantum degeneracy of the species (due to spin states, for example). The factor of \((2\pi)^3\) comes from Heisenberg’s uncertainty principle, which states that the product of the uncertainty of a particle’s position and momentum (in one dimension) must be at least \( 2\pi\hbar \). Hence, the uncertainty in a particle’s location in six-dimensional phase space is at least \((2\pi\hbar)^3\). This implies that the distribution function \( f_i \) has an inherent granularity — it actually gives the number of particles in a \((2\pi\hbar)^3\)-sized “pixel” of phase space. Hence, when we integrate over \( f_i \) we must make sure to integrate over the number of pixels. This accounts for the factor of \( 2\pi \) in the denominator for every component of the \( dp \) integral, after adopting the simplifying convention that \( \hbar = 1 \).

All that is left to be done then is to determine the form of the distribution function for the species. Typical distributions are the Fermi-Dirac distribution for particles with half-integer spin (like protons and electrons):

\[ f_{FD} = \frac{1}{e^{(E-\mu)/T} + 1} \]  

(2.2)

and the Bose-Einstein distribution for particles with integer spin (like photons):

\[ f_{BE} = \frac{1}{e^{(E-\mu)/T} - 1} \]  

(2.3)

where \( \mu \) is the chemical potential of the species. Note that these distributions do not depend on position or direction of momentum. Normally, the temperature factor in the exponent is given as \( k_B T \), where \( k_B \) is Boltzmann’s constant, which serves to convert temperature to energy. We again simplify our equations by setting this constant to unity.

### 2.1.1 Photons

The energy density of photons is denoted by \( \rho_r \).³ We can calculate the photon energy density directly using equation 2.1, substituting the Bose-Einstein distribution into the formula:

\[ \rho_r = g_i \int \frac{d^3p}{(2\pi)^3} \frac{E(p)}{e^{(E-\mu)/T} - 1} \]  

(2.4)

³Typically, \( \rho_r \) also includes the energy density of neutrinos. I will ignore neutrinos throughout this work, but the \( \rho_r \) I use does generalize to include neutrinos.
Note that, for photons, \( g_i = 2 \) (since they have two spin states), and \( E = p \). The chemical potential is negligible, so the above equation reduces to:

\[
\rho_i = \frac{1}{4\pi^3} \int \frac{p}{e^{p/T} - 1} \, d^3p = \frac{1}{4\pi^3} \int \frac{p}{e^{p/T} - 1} \, p^2 \, dp \, d\Omega
\]

\[
= \frac{1}{\pi^2} \int \frac{p^3}{e^{p/T} - 1} \, dp = \frac{T^4}{\pi^2} \int \frac{x^3}{e^x - 1} \, dx
\]

\[
= \frac{\pi^2}{15} T^4
\]

where on the first line, we switch the integration to spherical coordinates, and on the second line we introduce the dimensionless parameter \( x = p/T \). In the third line, we looked up the solution to the integral. Now, we have run ourselves into a corner by setting so many physical constants to unity. Equation 2.5 can be re-cast in SI units by adding in the appropriate factors of \( c, \hbar, \) and \( k_B \) such that the units match up. For reference, the SI values of these constants are as follows:

\[
c = 299,792,458 \text{ m/s} \quad (2.6)
\]

\[
\hbar = 1.05457148 \times 10^{-34} \text{ J} \cdot \text{s}
\]

\[
k_B = 1.3806503 \times 10^{-23} \text{ J/K}
\]

So, to convert \( K^4 \) to \( J/m^3 \), we must multiply by \( k_B^4 \) and divide by \( \hbar^3 c^3 \). Hence, in SI units, the photon energy density is given by:

\[
\rho_i = \frac{\pi^2}{15} \frac{k_B^4}{c^3 \hbar^3} T^4
\]

To find the current photon energy density, plug in the CMB temperature today, 2.725 K:

\[
\rho_{i,0} \approx 4.172 \times 10^{-14} \frac{\text{J}}{\text{m}^3} \quad (2.8)
\]

Let us now consider how the photon energy density evolves as a function of the scale factor in our expanding universe. Clearly, physical volumes scale as \( a^3 \). In addition, the energy of an individual photon is inversely proportional to its wavelength, which scales as \( a \) — think of the wave as being “stretched out” as the universe expands. Combining these effects, \( \rho_i \propto a^{-4} \). This gives us a formula for the photon energy density as a function of time:

\[
\rho_i(t) = \frac{\rho_{i,0}}{a^4(t)}
\]

Making use of equation 2.5, we can use the above relation to easily determine the evolution of the temperature of the photon field (the mean temperature of the CMB):

\[
T(t) = \frac{T_0}{a(t)}
\]
2.1.2 Baryons

First, recall that the term “baryons” is meant to refer to all luminous matter (the stuff that makes up stars, planets, and dust). We will denote the density of baryons as $\rho_b$. Baryon density cannot be calculated as easily as the photon density, since baryons do not behave as a gas with a well-defined temperature in today’s universe. However, we can make direct measurements of baryon density by counting up the matter we can see (from stars, dust clouds, galaxies, and galaxy clusters). CMB analysis provides an accurate measurement of baryon density. The three-year results from WMAP (Spergel et al., 2006) indicate that the present value of the baryon density, $\rho_{b,0}$, is:

$$\rho_{b,0} \approx 4.12 \times 10^{-28} \text{ kg m}^{-3} \approx 3.77 \times 10^{-11} \text{ J m}^{-3}$$  \hspace{1cm} (2.11)

which is equivalent to 1 proton per four cubic meters — an astonishingly small density, compared to densities we are used to on Earth. For example, the density of water (1 g/cm$^3$) is about $6 \times 10^{29}$ protons per cubic meter.

Comparing with equation 2.9, we see that at present, baryons are a significantly more dominant component of the universe than photons. However, the density of baryons makes up only about 4.2% of the total effective energy density, $\rho_0$. This requires that we search for more exotic forms of energy density to make up the difference.

2.1.3 Dark Matter

The earliest proposal that some sort of non-luminous matter, or dark matter, may be prevalent in the universe came in 1933 (Zwicky, 1933). Since that time, evidence from the rotation curves of galaxies ((Corbelli and Salucci, 2000), (Rubin et al., 1985)), in addition to other sources, has indicated that the total matter density is about five times larger than the density of luminous matter and, hence, dark matter must make up the difference. Cosmological models that include dark matter are called “CDM” models — which stands for cold dark matter, where “cold” is meant to mean “nonrelativistic”.

The density of dark matter is denoted as $\rho_c$, but typically, the dark matter density is inferred from the total matter density, $\rho_m = \rho_c + \rho_b$. The WMAP results (Spergel et al., 2006) give:

$$\rho_{m,0} \approx 2.39 \times 10^{-27} \text{ kg m}^{-3} \approx 2.14 \times 10^{-10} \text{ J m}^{-3}$$  \hspace{1cm} (2.12)

which makes up about 24% of the energy density of the universe.

Note that the matter density simply scales as $a^{-3}$, due to the scaling of physical volume. Hence:

$$\rho_m(t) = \frac{\rho_{m,0}}{a^3(t)}$$  \hspace{1cm} (2.13)

---

4We know this because the universe is observed to be flat, which requires the total energy density to take on a predictable value. This will be described in greater detail in §2.5.

5See Dodelson (2003), §2.4.3.

6This is in contrast to “HDM” models, which allow for relativistic dark matter — typically massive neutrinos.
Armed with equations 2.9 and 2.13, we can calculate the epoch of matter-radiation equality, by setting the densities of the two species equal. Adopting the subscript $r-m$ to denote this epoch, we have:

\[
\frac{\rho_{r,0}}{a^4(t_{r-m})} = \frac{\rho_{m,0}}{a^3(t_{r-m})}
\]

\[
a(t_{r-m}) = \frac{\rho_{\Lambda,0}}{\rho_{m,0}} \approx 1.95 \times 10^{-4}
\]

which, using equation 1.12, corresponds to redshift 5141. Before this time, radiation dominates the energy makeup of the universe, while after this time, matter dominates (until recent times when matter is eclipsed by a new form of energy, to be explored in the next section). Note that in my derivation, I have neglected the effect of neutrinos, which are typically included in $\rho_r$. If we had included neutrinos, we would have found $a_{\text{equiv}} \approx 3.27 \times 10^{-4}$ ($z \approx 3057$), which is on the same order as the result in equation 2.14. Comparing this to the redshift of recombination, $z_{\text{rec}} \approx 1100$, we see that by the time of recombination, the universe was well into the matter-dominated era.

### 2.1.4 Dark Energy

Exotic dark energy accounts for about 76% of the energy density of the universe at present. Dark energy, also called vacuum energy density, is thought to be energy density that is inherent to the vacuum, and is therefore of constant density in physical, rather than comoving, space. If dark energy density, denoted $\rho_\Lambda$, is closely related to Einstein’s cosmological constant, $\Lambda$, which he introduced to his equations of General Relativity to allow for a steady-state universe. Let us explore what the presence of such a vacuum energy density implies. The first law of thermodynamics dictates that $dE = -PdV$, where $dE$ is the change in energy of the species, $dV$ is the change in its volume, and $P$ is its pressure. For dark energy, if we let $dV$ be the change in physical volume due to the expansion of the universe, $dE = \rho_\Lambda dV$, since the energy density is constant. This implies that $P_\Lambda = -\rho_\Lambda$, meaning that the dark energy field actually has negative pressure.

We will prove in section §2.4.3 that such a negative pressure implies that the universe expands at an accelerating rate. Since $\rho_\Lambda$ is such a dominant component of the energy density today, we expect that the negative pressure from dark energy is causing accelerated expansion today. In fact, observational evidence for this acceleration ((Riess et al., 1998), (Schmidt et al., 1998)) was the motivation for including dark energy in the standard cosmological theory. The resulting cosmological theories incorporating dark energy are called “ΛCDM” models, as an extension of the previous cold dark matter models.

Now, let us calculate the epoch at which dark energy density and matter energy density are equivalent. Noting that, based on observation, $\rho_\Lambda \approx 6.83 \times 10^{-10}$ J/m$^3$, the scale factor of dark

---

7 In fact, what I have described here is the specific case where the equation of state, $w$, of the dark energy, is set to $-1$. See Dodelson (2003), §2.4.5, for more on this topic.

8 $\rho_\Lambda$ is constant in time as well as space.
energy-matter equivalence is given by:

\[ a(t_{\Lambda}) = \left( \frac{\rho_{m,0}}{\rho_\Lambda} \right)^{1/3} = 0.68 \] (2.15)

The redshift of equivalence is 0.47 — so this equivalence epoch occurred fairly recently. Today, we live in a \( \Lambda \)-dominated universe that has recently begun to expand at an accelerating rate.

\section*{2.2 The Metric}

In the familiar Cartesian coordinate system, the distance between two points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) is simply \(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\). In terms of infinitesimal coordinate displacements \(dx, dy, \text{ and } dz\), the infinitesimal distance \(ds\) is given by \(ds^2 = dx^2 + dy^2 + dz^2\). If we switch our coordinate system to spherical coordinates,\(^9\) and consider infinitesimal coordinate displacements \(dr, d\phi, \text{ and } d\theta\), the distance is now given by \(ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi\). The “coefficients” required to perform this map from coordinate displacements to physical distances are collectively called the metric for the coordinate system.

We denote a position vector in a generic coordinate system as \(x^i\), where the superscript is an index that ranges over the basis coordinates of the system.\(^10\) With this notation, we can consider an infinitesimal displacement \(dx^i\) in this coordinate system. The physical line element\(^11\) \(ds^2\) subtended by such a displacement is determined by the metric:

\[ ds^2 = \sum_{ij} g_{ij} dx^i dx^j \] (2.16)

where we have denoted the components of the metric as \(g_{ij}\).\(^12\) Note that in Euclidean space, the physical distance between two points is invariant — it is the same regardless of what coordinate system (Cartesian, spherical) is used to label the positions.

\subsection*{2.2.1 Four-vectors and Special Relativity}

In special relativity, the physical distance between two points is decidedly not invariant — if observers traveling at different speeds measure the length of the exact same object, they will arrive at different results. However, if we expand our notion of “position” to include time as well, and hence allow our three-dimensional position vectors to become four-dimensional vectors that refer

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\(^9\)As before, I use \(\phi\) as the azimuthal angle and \(\theta\) as the polar angle.

\(^10\)For example, in three dimensional spherical coordinates, we could let \(i\) range over \(r, \theta, \text{ and } \phi\), so that we would identify the triple \((r, \theta, \phi)\) with \((x^r, x^\theta, x^\phi)\). Typically, letters (like \(\mu, \nu, i, \text{ or } j\)) in the superscript denote a vector, while integer values \((0, 1, 2, \ldots)\) denote specific components.

\(^11\)In this context, the line element is the squared physical distance, but line element is a more general term, which we will use for \(ds^2\) from now on.

\(^12\)Note that the metric allows for cross-components. The metrics for Cartesian and polar coordinate system only have non-zero diagonal components, where \(i = j\).
to specific events in spacetime, we recall that the following quantity is invariant:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

(2.17)

where $ds^2$ is not simply a coincidental choice; it is in fact the line element of special relativity.

When working with four-vectors, we adopt a few new notational conventions. A four-vector will be denoted by a Greek index that implicitly ranges over $(0, 1, 2, 3)$ where the zero-component is the time component, and the others are the three spatial coordinates. A Latin index implicitly ranges over $(1, 2, 3)$, so it denotes the spatial component of a four-vector. Hence, we could write a typical four-vector as $x^\mu = (x^0, x^i)$ in keeping with these conventions. We now re-write the general line element (equation 2.16) for a generic infinitesimal four-vector displacement:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

(2.18)

In this form, we have also adopted the Einstein summation convention, whereby repeated indices are implicitly summed over. Hence, the right side of equation 2.18 has an implicit double sum over $\mu$ and $\nu$ from 0 to 3.

We now see that equation 2.17 is a special case of equation 2.18, with $g_{\mu\nu}$ set to the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

(2.19)

which defines the physics of objects in flat spacetime. Recall the interpretation assigned to $ds^2$ in special relativity: $ds^2 < 0$ indicates a timelike worldline and $\sqrt{|ds^2|}$ is the proper time measured by an observer moving from one event to the other; $ds^2 > 0$ indicates a spacelike worldline where $ds$ is the proper distance between the events, as measured in a frame where the events occur simultaneously. This interpretation holds not only for special relativity, but carries over to the generic line element in equation 2.18.

### 2.2.2 The Metric in an Expanding Universe

We can describe the expanding universe introduced in the previous chapter using the metric formulation we just developed. Recall from equation 1.2 that the physical distance is related to the comoving distance by the scale factor: $a^2(t) \sum (x^i)^2 = \sum (r^i)^2$, where $x^i$ is the comoving displacement vector and $r^i$ is the physical displacement vector. Hence, the only modification we have to

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13This is true due to the principle that the speed of light is constant in all reference frames. See Schutz (Schutz, 1985) §1.6 for a proof.
14These conventions are fairly standardized.
15From Peebles (1993), §2.
make to the Minkowski metric is to scale the position displacements by $a^2(t)$:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \tag{2.20}$$

This is the Friedmann-Robertson-Walker (FRW) metric for a flat, isotropic, and homogenous universe. Note that when using this metric, our four-vectors must use Cartesian comoving spatial coordinates and physical (not conformal) time. This is the standard coordinate system that we will use throughout, unless noted otherwise.

### 2.2.3 A WORD ON CURVATURE

We can rewrite the line element of the flat FRW model in spherical coordinates:

$$ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\theta + r^2 \sin^2 \theta d\phi \right) = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega \right) \tag{2.21}$$

This generalizes this to an isotropic, homogenous universe with arbitrary curvature as follows:

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1-Kr^2} + r^2 d\Omega \right) \tag{2.22}$$

where the curvature, $K$, with units of inverse length squared, determines the amount by which space deviates from flatness.\(^{16}\) Clearly, when $K = 0$, equation 2.22 reduces to the line element for a flat universe. When $K > 0$, space is positively curved, or closed (in this case, we can think of space as the three-dimensional surface of a four-dimensional sphere — it has no boundaries, but a finite volume). In the negative curvature, $K < 0$, case, space is open (and we can think of space as the three-dimensional analog of a hyperboloid). Observation shows that space is remarkably close to flat, so for most of this text, I assume that $K = 0$. This assumption simplifies the equations dramatically. Where appropriate, I will cite the corresponding curved-universe equivalents of derived results.

### 2.3 A MATHEMATICAL INTERLUDE

Before moving on, we must pause and describe some of the mathematical tools necessary to work in a four-dimensional spacetime with an arbitrary metric. First, a notational clarification. The notation $x^\mu$ can be used both to denote a specific set of coordinates and to denote a “position” vector in that coordinate system. For example, the set $x^\mu = (t, x^i)$ denotes our standard time/comoving Cartesian coordinate system.\(^ {17}\) The usage of $x^\mu$ should be clear from context — for example, when a derivative

---

\(^{16}\) A derivation and thorough explanation of equation 2.22 is given in Hartle (2003), §18.6.

\(^{17}\) I am glossing over the distinction between the expression of coordinates in a coordinate basis versus an orthonormal basis. See Hartle (2003), §7.8; or Schutz (1985), §5.6.
is taken with respect to $x^\nu$, the coordinate system $x^\nu$ is implied.

### 2.3.1 Vectors, One-Forms, and Tensors

We have already met vectors, such as $v^\mu$, which define events in spacetime. The associated one-form, $v_\mu$, is a function that takes a vector as its argument and returns a scalar. Contracting a vector with its dual one form gives an invariant: $v_\mu v^\mu$. A vector and its dual one-form are related by the metric:

$$v_\mu = g_{\mu \nu} v^\nu$$
$$v^\mu = g^{\mu \nu} v_\nu$$

(2.23)

This technique is called “index raising” or “index lowering”. Note that a lower index denotes a one-form, while an upper index denotes a vector. In Euclidean three-space, a vector is equivalent to its associated one-form, since the metric is the identity matrix. This is why it is natural to consider the “dot product” of two vectors in ordinary Euclidean space. In reality, the more general inner product is the contraction of a vector $v^\mu$ with its associated one form, $v_\mu v^\nu = g_{\mu \nu} v^\nu v^\mu$. In the Euclidean case, the metric is just the identity matrix, so there is no need to consider the distinction between one-forms and vectors. In an arbitrary space, we need to remember to include these extra factors of the metric.

Equation 2.23 should raise questions about the nature of the metric. We have already seen that it can serve as a function that takes two vectors as its arguments and returns a scalar, as in equation 2.18. However, in equation 2.23, we see that the metric can also act as a function of a single vector that returns a one-form. The upper-indexed version of the metric, $g^{\mu \nu}$, is a function that takes a one-form and returns a vector.

We now make the definition that an $(M^N)$ tensor is a function that takes $N$ vectors and $M$ one-forms and returns a scalar. Hence, a one-form is a $(1^0)$ tensor, a vector is a $(0^1)$ tensor, and the metric is a $(0^2)$ tensor. The number of lower indices on a tensor indicates its value of $N$, or its covariant rank, while the number of upper indices indicates its value of $M$, its contravariant rank. Hence, by this convention, $g^{\mu \nu}$ is a $(0^2)$ tensor. To generalize further, we see that contracting a $(M^N)$ tensor with a $(P^Q)$ tensor yields a $(M-N-P)$ tensor.

The two forms of the metric, $g_{\mu \nu}$ and $g^{\mu \nu}$ are privileged in the coordinate system because they have the ability to switch the nature of an index (from contravariant to covariant and vice versa, respectively). This index raising procedure works on general tensors just as it does on vectors and one-forms. Let us apply the index-raising procedure to the metric itself:

$$g^{\mu \nu} = g^{\mu \alpha} g^{\nu \beta} g_{\alpha \beta} = g^{\mu \alpha} g^{\nu \alpha}$$

(2.24)

In the event that the dummy index $\alpha$ equals $\nu$, we have $g^{\mu \nu} g_{\nu}$ on the left hand side of the equation. If $g_{\nu} = 1$ and all other components of $g_{\alpha} = 0$, the equation is satisfied. Hence,

$$g^{\nu \nu} g_{\nu} = g^{\nu} = \delta^{\nu}$$

(2.25)

---

18 This extremely brief summary follows Schutz (1985), §3. The source offers much more detail as to the nature of one-forms and generic tensors, in addition to offering proofs of some of the assertions in this section.

19 This explanation is adapted from Dodelson (2003), §2.1.1.
where $\delta_\alpha^\nu$ is the Kronecker delta. If we think of these objects as matrices, we see that the $(_2^0)$ and $(^0_2)$ forms of the metric are in fact matrix inverses of each other. This is a unique property of the metric in a given coordinate system, and makes finding the contravariant form of the metric a fairly simple matter.

### 2.3.2 Particle Motion

In our four-dimensional spacetime, we must specify the position of a particle with a four-vector, $x^\mu$. In classical physics, we can parameterize the motion of the particle through space as a function of time (so we can speak of the particle’s position evolving in time). In relativity, time is one of the dependent variables in our four-vector, and hence we introduce the affine parameter, $\lambda$, to parameterize the motion of a particle through four-dimensional spacetime. The affine parameter monotonically increases as the particle moves from its starting position in spacetime to its final position. We therefore consider the position of our particle to be a function of this affine parameter, $x^\mu(\lambda)$.

In the theory of General Relativity, the curvature of spacetime is all that drives the motion of particles — they simply follow geodesics.\(^{20}\) The geodesic is given by the aptly named geodesic equation:

$$\frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\delta} \frac{dx^\alpha}{d\lambda} \frac{dx^\delta}{d\lambda}$$  \hspace{1cm} (2.26)

where we have defined the Christoffel symbol, $\Gamma^\mu_{\alpha\delta}$, to be the following $(^1_2)$ tensor combination of the metric and its derivatives:

$$\Gamma^\mu_{\alpha\delta} = \frac{1}{2} g^{\gamma\mu} \left( g_{\gamma\alpha,\delta} + g_{\gamma\delta,\alpha} - g_{\alpha\delta,\gamma} \right)$$  \hspace{1cm} (2.27)

Note that the Christoffel symbol is symmetric in its covariant indices: $\Gamma^\mu_{\alpha\delta} = \Gamma^\mu_{\delta\alpha}$. The Christoffel symbol is a measure of how the metric is changing at a given point in spacetime. In Euclidean space, and in Minkowski spacetime, the Christoffel symbol is always zero, and hence the geodesic equation reduces to $d^2 x^\mu / d\lambda^2 = 0$, which is just a statement of Newton’s first law of motion.

We can make an implicit definition of the affine parameter via the particle’s four-momentum, which, as in special relativity, is given by $P^\mu = (E, P^i)$, where $E$ is the energy of the particle, and $P^i$ is its three-momentum (measured in comoving coordinates, of course). This implicit definition is given by:

$$P^\mu = \frac{dx^\mu}{d\lambda}$$  \hspace{1cm} (2.28)

\(^{20}\)This of course assumes the absence of other forces. General Relativity removes the concept of gravity as a “force” and explains it in terms of spacetime curvature. This does not completely eliminate force, however — we still consider electromagnetism to be an external force on particles, for example.
eliminate the affine parameter:

\[
\frac{d}{d\lambda} = \frac{dx^0}{d\lambda} \frac{d}{dt} = E \frac{d}{dt}
\]  

(2.29)

where in the second equality, we noted that, by our definition, \(dx^0/d\lambda = P^0 = E\).

This allows us to rewrite the geodesic equation in a more user-friendly form. First note that the left-hand side of equation 2.26 is given by:

\[
\frac{d^2 x^\mu}{d\lambda^2} = \frac{d}{d\lambda} \left( \frac{d x^\mu}{d\lambda} \right) = P^\mu \frac{d}{dt} \left( P^\mu \right) = P^\mu \frac{dP^\mu}{dt}
\]  

(2.30)

Now, substituting in the definition of equation 2.28 to the right-hand side of equation 2.26, we have our final form of the geodesic equation:

\[
\frac{dP^\mu}{dt} = -\Gamma^\mu_{\alpha\delta} P^\alpha P^\delta
\]  

(2.31)

### 2.3.3 Application to the FRW Metric

Before concluding our mathematical aside, we calculate the Christoffel symbols for a flat FRW metric (equation 2.20), as they will be necessary for the physical calculations of the following sections.

For reference, the covariant (upper-indexed) version of the flat FRW metric is given by:

\[
g^{00} = -1 \quad g^{ii} = \frac{1}{a^2(t)}
\]  

(2.32)

with all other components equal to zero.

Using equation 2.27 as the starting point, let us begin by first noting that the only nonzero components of the sum on the right-hand side are those with \(\gamma = \mu\), since \(g^{\gamma\mu}\) is diagonal:

\[
\Gamma^\mu_{\alpha\delta} = \frac{1}{2} g^{\mu\nu} \left( g_{\nu\alpha,\delta} + g_{\nu\delta,\alpha} - g_{\alpha\delta,\nu} \right)
\]  

(2.33)

Let us first consider the \(\mu = 0\) case. Note that \(g_{0\alpha}\) and \(g_{0\delta}\) vanish unless \(\alpha = 0\) or \(\delta = 0\). However, \(g_{00} = -1\) is constant, and hence all its derivatives are zero. Hence, the first two terms are zero:

\[
\Gamma^0_{a\delta} = \frac{1}{2} \left( g_{a\delta,0} \right)
\]  

(2.34)

where I have substituted in the value of \(g^{00}\). Now, note that the right-hand side is nonzero only when \(\alpha = \delta \neq 0\). Hence, the \(\mu = 0\) component is given by:

\[
\Gamma^\mu_{\alpha\delta} = \frac{1}{2} \left( \frac{d}{dt} [a^2(t)] \right) = a(t) \frac{da}{dt} = a^2(t) H(t)
\]  

(2.35)

Turning our attention to the \(\mu = i\) components, we note that the third term on the right-hand side of equation 2.33 will vanish, since none of the components of the metric have position depen-
\[ \Gamma_{\alpha\delta} = \frac{1}{2a^2} (g_{\alpha,\delta} + g_{\delta,\alpha}) \]  
(2.36)

Clearly, one of \( \alpha \) or \( \delta \) must equal \( i \) for the right-hand side to be nonzero. In addition, the other index must be zero, since spatial derivatives of the metric vanish. Taking advantage of the symmetry of the Christoffel symbol, we are left with:

\[ \Gamma_{00} = \Gamma_{0i} = \frac{1}{2a^2} (g_{0i,0} + g_{00,i}) = \frac{1}{a(t)} \frac{da}{dt} = H(t) \]  
(2.37)

In summary, the only nonzero components of the Christoffel symbol in the flat FRW universe are as follows:

\[
\begin{array}{c|c}
\Gamma_{00} & a \frac{da}{dt} = a^2 H \\
\Gamma_{0i} & \frac{da}{dt} / a = \Gamma_{0i} = H(t) \\
\end{array}
\]

(2.38)

### 2.4 The Einstein Equation

We now have almost all the mathematical and physical background required to tackle the full formalism of Einstein’s General Relativity. We are only missing one more piece of the puzzle, which I introduce in the next section.

#### 2.4.1 The Stress-Energy Tensor

We have already considered the energy density of a homogenous universe. How do we account for momentum as well? The answer comes in the form of the stress-energy tensor (alternatively, the energy-momentum tensor), which completely characterizes the energy and momentum content of a fluid. The tensor, which we denote as \( T^{\mu\nu} \), is defined as the flux of \( \mu \) momentum (\( P^\mu \)) across a surface of constant \( x^\nu \) (Schutz, 1985). Taking into account our definition of four-momentum, we can make the following conclusions about the components of the stress-energy tensor:

\[
\begin{align*}
T^{00} & = \text{energy density} \\
T^{0i} & = \text{energy flux in the } i \text{ direction} \\
T^{i0} & = \text{i momentum density} \\
T^{ij} & = \text{flux of i momentum in the } j \text{ direction}
\end{align*}
\]

The stress-energy tensor is symmetric, \( T^{\mu\nu} = T^{\nu\mu} \). In the homogenous FRW universe, we consider the entire universe to be a perfect isotropic fluid. In this case, the stress-energy tensor is most easily
understood in the following form:

\[ T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \] (2.40)

where \( \rho \) is the energy density and \( P \) is the pressure of the fluid.

2.4.2 THE FIELD EQUATIONS

Einstein determined how the metric and the energy-momentum tensor interact with each other. Since we have now met both objects, we can state the Einstein Field Equations that describe their evolution:

\[ G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2} g^\mu_\nu R = 8\pi G T^\mu_\nu \] (2.41)

where we have defined the Einstein tensor, \( G^\mu_\nu \), as the “geometric” component of the Einstein equation. The “matter” or “energy” component is the stress-energy tensor \( T^\mu_\nu \). Hence, in this form, the Einstein equation clearly represents the interaction of matter and spatial curvature. Note that the Einstein equation is a second-rank tensor equation, so it actually represents a system of sixteen equations. However, the Einstein tensor and the stress-energy tensor are symmetric, so there are only ten independent equations.

The Einstein tensor is defined in terms of the Ricci tensor, \( R^\mu_\nu \), which is named in honor of Italian mathematician Gregorio Ricci-Curbastro (1853-1925). It is given by the following combination of the metric and its derivatives, most concisely expressed in terms of Christoffel symbols:

\[ R^\mu_\nu = \Gamma^a_{\nu\mu} - \Gamma^a_{\mu\nu} + \Gamma^a_{\beta\gamma} \Gamma^\beta_{\mu\nu} - \Gamma^a_{\gamma\nu} \Gamma^\beta_{\mu\beta} \] (2.42)

The Ricci scalar, \( R \), is the contraction of the Ricci tensor with the metric:

\[ R = g^{\mu\nu} R_{\mu\nu} \] (2.43)

The Ricci tensor and scalar describe the local curvature of space. The Ricci scalar is the multi-dimensional analog of Gaussian (scalar) curvature. In two dimensions, at a point where \( R > 0 \), the area of a circle is less than \( 2\pi R \), and where \( R < 0 \), a circle has area greater than \( 2\pi R \). In more than two dimensions, the scalar curvature is not sufficient to describe the curvature of a manifold, and the full Ricci tensor is required.

In the following section, we apply the Einstein equation to the FRW universe to derive important fundamental results about the interplay of energy density, pressure, and scale factor.
2.4.3 Application to the FRW Universe

The time-time component of equation 2.41 describes the evolution of the scale factor in the FRW universe. However, we still need to calculate the entire Ricci tensor so that we can form the Ricci scalar, which is needed to compute the Einstein tensor. We already know the Christoffel symbols in the FRW metric (equation 2.38), so we can plug them into equation 2.42. For the remainder of this section, I will denote derivatives with respect to physical time by an overdot, and I will keep the time dependence of the scale factor implicit.

Let us begin with the time-time component:

\[ R_{00} = \Gamma_{00,a}^a - \Gamma_{0a,0}^a + \Gamma_{a0}^a \Gamma_{00}^\beta - \Gamma_{\beta0}^\beta \Gamma_{0a}^a \]  

(2.44)

The Christoffels with zero-zero in their lower indices vanish, so the first and third terms drop immediately. Also note that the other terms will generate nonzero terms only when the spatial coordinates are considered:

\[ R_{00} = -\Gamma_{0i,0}^i - \Gamma_{i0}^i \]  

(2.45)

\[ = -3 \left( \frac{d}{dt} \frac{\dot{a}}{a} \right) - 3 \left( \frac{\dot{a}}{a} \right)^2 \]

\[ = -3 \left( -\frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \right) - 3 \left( \frac{\dot{a}}{a} \right)^2 \]

\[ = -3 \frac{\ddot{a}}{a} \]

Moving on to the time-space component (symmetry makes this valid for the space-time component as well), we have:

\[ R_{0i} = \Gamma_{0i,a}^a - \Gamma_{0a,i}^a + \Gamma_{\beta a}^a \Gamma_{0i}^\beta - \Gamma_{\beta i}^\beta \Gamma_{0a}^a \]  

(2.46)

The first two terms reduce to \( \Gamma_{0i,j}^j \) and \( \Gamma_{0j,i}^j \), respectively. Both vanish since there is no spatial dependence in the Christoffel symbols. In the third term, we note that \( \Gamma_{\beta i}^\beta \) is nonzero only when \( \beta = i \). However, in this case, \( \Gamma_{\beta a}^a \) is always zero and the third term vanishes. The fourth term also vanishes, because \( \Gamma_{\beta a}^a \) requires that \( \alpha \) and \( \beta \) be spatial, but then the first component of the term has three spatial indices, and all such Christoffel symbols are zero. Hence:

\[ R_{0i} = R_{i0} = 0 \]  

(2.47)

Finally, consider the space-space component:

\[ R_{ij} = \Gamma_{ij,a}^a - \Gamma_{ia,j}^a + \Gamma_{\beta a}^a \Gamma_{ij}^\beta - \Gamma_{\beta j}^\beta \Gamma_{ia}^a \]  

(2.48)

Note that the first term is nonzero only when \( \alpha = 0 \), the second term vanishes due to the spatial derivative, and the third term is nonzero only when \( \beta = 0 \). Splitting up the fourth term into the
time and space components of $\beta$, we have:

$$R_{ij} = \Gamma^0_{i,j} + \Gamma^0_{0,i} - \Gamma^0_{j,0} - \Gamma^0_{i,0}$$

(2.49)

Clearly, the first and second terms here are nonzero only when $i = j$. In the third term, the first component is nonzero only when $\alpha = j$, and the second component is nonzero only when $\alpha = i$, and therefore the third term is nonzero only when $i = j$. In the fourth term, first note that only the $\alpha = 0$ element of the sum has a chance to generate nonzero elements. Now, the first component requires $k = j$ and the second component requires $k = i$. Hence, the space-space component is nonzero only when $i = j$:

$$R_{ii} = \Gamma^0_{i,i} + \Gamma^0_{i,0} - 2\Gamma^0_{i,0}$$

(2.50)

$$= \frac{d}{dt} (\alpha \dot{\alpha}) + 3 \left( \frac{\alpha}{a} \right) (a^2) - 2 \left( \frac{\dot{\alpha}}{a} \right) (a \ddot{\alpha})$$

$$= (\dot{a}^2 + a \ddot{a}) + \dot{a}^2$$

$$= 2\dot{a}^2 + a \ddot{a}$$

In summary, the Ricci tensor for the flat FRW universe is given by:

$$R_{\mu\nu} = \begin{pmatrix}
-3\ddot{a}/a & 0 & 0 & 0 \\
0 & 2\dot{a}^2 + a \ddot{a} & 0 & 0 \\
0 & 0 & 2\dot{a}^2 + a \ddot{a} & 0 \\
0 & 0 & 0 & 2\dot{a}^2 + a \ddot{a}
\end{pmatrix}$$

(2.51)

It is now a simple matter to form the Ricci scalar:

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}$$

(2.52)

$$= 3 \frac{\ddot{a}}{a} + 3 \left( \frac{1}{a^2} [2\dot{a}^2 + a \ddot{a}] \right)$$

$$= 6 \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)$$

Let us now consider the time-time component of the Einstein equation:

$$G_{00} = R_{00} - \frac{1}{2} g_{00} \mathcal{R} = 8\pi G T_{00}$$

(2.53)

Plugging in everything that we have just derived, and recalling that $T_{00}$ is simply the total energy density, $\rho$, we have:

$$-3 \frac{\ddot{a}}{a} + 3 \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = 8\pi G \rho$$

(2.54)

Cleaning up a bit, we have finally arrived at the Friedmann equation for the evolution of the scale
factor in a flat FRW universe:

\[ H^2 - \frac{8\pi G}{3} \rho = 0 \] (2.55)

Since we have derived the full Ricci tensor, we can also derive the space-space component of the Einstein equation in the FRW universe. We speak of the space-space component because all the diagonal space-space elements of \( R_{\mu\nu} \), \( \tilde{g}_{\mu\nu} \), and \( T_{\mu\nu} \) are equal. Beginning with equation 2.41, we have:

\[
R_{ii} - \frac{1}{2}\tilde{g}_{ii}R = 8\pi G T_{ii}
\]

\[ 2\dot{a}^2 + a\ddot{a} - 3a^2 \left( \frac{\dot{a}}{a} + \frac{\ddot{a}}{a^2} \right) = 8\pi G (a^2 \mathcal{P}) \]

where we remembered a factor of \( \tilde{g}_{ii} \) to lower the index of \( T_i^j = \mathcal{P} \). Cleaning this up, we have:

\[
\frac{\ddot{a}}{a} + \frac{1}{2} H^2 = -4\pi G \mathcal{P}
\] (2.57)

We can combine equations 2.55 and 2.57 to arrive at an equation describing the second-order behavior of the scale factor:

\[
\frac{\ddot{a}}{a} = -4\pi G \left( \mathcal{P} + \frac{\rho}{3} \right)
\] (2.58)

Now we see why a negative pressure drives accelerated expansion, as we claimed in §2.1.3. If \( \mathcal{P} < -\rho/3 \), equation 2.58 forces \( \ddot{a} > 0 \), and the scale factor increases at an accelerating rate.

### 2.5 Rates, Times and Distances in the FRW Universe

Armed with the Friedmann equation, we now calculate several useful quantities in the isotropic, homogenous universe. Before we begin, note that if we had performed the derivation of the previous section using the more general metric of equation 2.22, which allows for curvature, we would have arrived at the following more general form of the Friedmann equation:

\[
H^2(t) - \frac{8\pi G}{3} \rho(t) = -\frac{K}{a^2(t)}
\] (2.59)

Where \( K \) is the curvature parameter that was introduced in equation 2.22. We will work with this slightly more general form in this section, since it does not add much complexity to the calculations.

#### 2.5.1 Evolution of the Hubble Rate

Our first task is to unpack the Friedmann equation into a more useful and physically intuitive form. Evaluating equation 2.59 at \( t = t_0 \), we arrive at the following relationship:

\[
K = \frac{8\pi G}{3} \rho_0 - H_0^2
\] (2.60)
from this we can solve for the density today that forces the curvature parameter \( K \) to go to zero, called the critical density, \( \rho_{cr,0} \):

\[
\rho_{cr,0} = \frac{3H_0^2}{8\pi G} \approx 1 \times 10^{-26} \text{ kg m}^{-3} \approx 9 \times 10^{-10} \text{ J m}^{-3}
\]  

(2.61)

where I substituted \( H_0 = 72 \text{ km/s/Mpc} \) — the best-fit WMAP value (Spergel et al., 2006). This density is equivalent to roughly six protons per cubic meter.

Note that if \( \rho_0 > \rho_{cr,0} \), we have positive curvature and a closed universe, while if \( \rho_0 < \rho_{cr,0} \), the universe is open. Using this notation, we can substitute:

\[
K = \frac{8\pi G}{3} [\rho_0 - \rho_{cr,0}]
\]  

(2.62)

into equation 2.59 to obtain another formulation of the Friedmann Equation:

\[
H^2(t) = \frac{8\pi G}{3} \left[ \rho(t) + \frac{\rho_{cr,0} - \rho_0}{a^2(t)} \right]
\]  

(2.63)

Now, let us define the curvature density \( \rho_{K,0} \equiv \rho_{cr,0} - \rho_0 \) as the “missing” density required to make the universe flat. Also, we split up the density \( \rho(t) \) into its radiation, matter, and dark energy components, noting how the separate components evolve in time (as we derived in §2.1):

\[
H^2(t) = \frac{8\pi G}{3} \left[ \rho_{r,0} + \rho_{m,0} + \rho_{K,0} + \rho_{\Lambda} \right]
\]  

(2.64)

It is common practice to express the component densities as fractions of the critical density. We adopt the standard notation \( \Omega_{\text{species}} \equiv \rho_{\text{species}} / \rho_{cr} \) for this purpose. Switching to this notation, we have:

\[
H^2(a) = \frac{8\pi G}{3\rho_{cr}} \left[ \Omega_{r,0} + \Omega_{m,0} + \Omega_{K,0} + \Omega_{\Lambda} \right]
\]  

(2.65)

where we have also allowed the equation to be an explicit function of the scale factor. Finally, use equation 2.61 to simplify, and we are left with a very simple form for the Hubble rate:

\[
H(a) = H_0 \sqrt{\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{K,0}}{a^2} + \Omega_{\Lambda}}
\]  

(2.66)

This form of the Friedmann equation is very useful, since it allows us to calculate the Hubble rate at an arbitrary scale factor as a function of observable quantities. We will make use of it in the next sections to improve our understanding of time and distance in the FRW universe.
2.5.2 Time Calculations

Let us re-cast equation 1.6 so that we can find the conformal time as a function of the scale factor, as opposed to physical time:

\[
\tau(a) = \int_0^a \frac{dt}{a'(a')} = \int_0^a \frac{1}{a'^2(a')} \frac{d\tau}{da'} \tag{2.67}
\]

In effect, all I have done is “multiply” the integrand of equation 1.6 by \( \frac{da'}{da} \). Now, let us recast equation 1.4 as a function of the scale factor:

\[
\frac{1}{aH(a)} = \frac{dt}{da(a)} \tag{2.68}
\]

Plugging this into equation 2.67, we have:

\[
\tau(a) = \int_0^a \frac{1}{a'^2 H(a')} \cdot da' \tag{2.69}
\]

Making use of the final form of the Friedmann equation (2.66), we have:

\[
\tau(a) = \int_0^a \frac{1}{H_0 \sqrt{\Omega_{r,0} + a^4 \Omega_{m,0} + a^2 \Omega_{K,0} + a^4 \Omega_{\Lambda}}} \cdot da' \tag{2.70}
\]

Equation 2.70 gives us the ability to calculate conformal time as a function of scale factor and observable quantities. Identifying the integrand of equation 2.70 as \( \frac{d\tau}{da} \) and noting that \( d\tau = \frac{dt}{a} \), we can also use this form to easily calculate the physical time:

\[
t(a) = \int_0^a \frac{1}{a'^2 H_0 \sqrt{\Omega_{r,0} + a^4 \Omega_{m,0} + a^2 \Omega_{K,0} + a^4 \Omega_{\Lambda}}} \cdot da' \tag{2.71}
\]

These formulae for time are very useful in the numerical methods that model the evolution of the universe. They will also allow us to calculate time (physical and conformal) out into the future, since there is nothing in these equations preventing us from integrating out to a scale factor greater than one. The equations are valid for all positive choices of \( a \).

2.5.3 Distance Calculations

Our final goal is to use our new time formulae to calculate the angular scale of the horizon at last scattering as seen today. This will justify the existence of the horizon problem, which we discussed in §1.4. In order to do this in a flat universe, we need to calculate the conformal time of recombination, \( \tau_{\text{rec}} \), and the conformal time today, \( \tau_0 \). Then, interpreting the comoving times as conformal distances, the angular size of the last-scattering horizon is given by \( \theta_{\text{horizon}} = \tau_{\text{rec}} / (\tau_0 - \tau_{\text{rec}}) \).

We numerically integrate equation 2.70 from \( a = 0 \) to \( a = a_{\text{rec}} \approx 0.001 \) to get \( \tau_{\text{rec}} \) and from \( a = 0 \) to \( a = a_0 = 1 \) to get \( \tau_0 \). For the parameters, use the WMAP-best fit values for a flat \( \Lambda \text{CDM} \)
The universe:\footnote{We have already met all of these parameters, I am just casting them in the form that is required for equation 2.70. The radiation fraction $\Omega_r$ includes neutrinos here.}  

\[
H_0 = 72 \frac{\text{km}}{\text{s-Mpc}} = 2.44 \times 10^{-4} \text{Mpc}^{-1} 
\]

\[
\Omega_{r,0} = 7.79 \times 10^{-5} 
\]

\[
\Omega_{m,0} = 0.238 
\]

\[
\Omega_{\Lambda,0} = 0 
\]

\[
\Omega_{\Lambda} = 0.762 
\]

Plugging these values into equation 2.70, we find $\tau_0 \approx 14600 \text{Mpc}$ and $\tau_{\text{rec}} \approx 300 \text{Mpc}$. Using these values, we see that $\theta_{\text{horizon}} \approx 1.2^\circ$. CMB photons that arrive at angular separations greater than $1.2^\circ$ would not have been in causal contact before recombination if it were not for the inflationary epoch.
In this chapter, we develop a first-order perturbation theory based on the homogenous FRW universe we have worked with thus far. This theory allows for slight deviations from isotropy that vary in space and time. To derive the evolution of the universe in this framework, we need to bring to bear both Einstein’s field equations and Boltzmann’s equations for fluid dynamics. Since the different components of the universe (photons, neutrinos, baryons, and dark matter) have different characteristics, we need to consider the evolution of each separately, ultimately yielding a coupled system of equations for their evolution. To conserve space, we will only explicitly derive the Boltzmann equation for photons in this chapter — this is the equation which forms the basis for the actual CMB spectrum calculations which we describe in the next chapter. We will simply cite the final equations for the evolution of the other components when we have finished with the photons.

3.1 Preliminaries

The first step in developing a perturbation theory is deciding on a metric, which allows us to describe geometry, and therefore gravity. The metric we choose should be based on slight deviations from the FRW metric (equation 2.20), allowing for slight local deviations from perfect isotropy and homogeneity. In this section, we describe such a metric and calculate the relevant Christoffel symbols.

3.1.1 The Metric

Our choice for the metric in our perturbed universe is:

\[
\begin{pmatrix}
-1 - 2\Psi(x^\mu) & 0 & 0 & 0 \\
0 & a^2(t)(1 - 2\Phi(x^\mu)) & 0 & 0 \\
0 & 0 & a^2(t)(1 - 2\Phi(x^\mu)) & 0 \\
0 & 0 & 0 & a^2(t)(1 - 2\Phi(x^\mu))
\end{pmatrix}
\] (3.1)

Where the perturbation parameter \(\Psi\) corresponds to a “Newtonian” gravitational potential, and \(\Phi\) corresponds to the local curvature of spacetime. With these sign conventions, an overdense region will have \(\Psi, \Phi < 0\). These parameters are taken to always be small — meaning that \(\Psi, \Phi \ll 1\) at all times and over all space — allowing us to drop all terms of higher than linear order in them.

\(^1\)Note that I follow the sign convention of equation 5 of Ma and Bertschinger (1995), not that of equation 4.9 of Dodelson (2003). Note also that Ma and Bertschinger define \(g_{00} = -a^2(1 + 2\Psi)\) — this is due to the fact that they use conformal time as \(x^0\), whereas I (following Dodelson) use physical time.
In addition, to conserve space, I will often allow the dependence on space and time \( x^\mu \) to remain implicit for these perturbations, along with the derived quantities.

Before proceeding, we should take note of the assumptions we are making by using equation 3.1 as our metric. First, note that we have a freedom to choose a gauge in which to express our perturbation. We have chosen the conformal Newtonian gauge, but we should be aware that other gauges, such as the synchronous gauge, also exist. The following derivation can be performed in a gauge-invariant form (Bardeen, 1980), and it is possible to move from gauge to gauge (Ma and Bertschinger, 1995). Second, note that the metric of equation 3.1 contains only scalar perturbations. Additional perturbations, called vector and tensor perturbations, are also allowed. We can safely ignore them, because they are completely decoupled from the scalar perturbations (at first order), and have a small effect for our purposes.

### 3.1.2 The Christoffel Symbol

We will need the time component of the Christoffel symbol in our derivation of the Boltzmann equation for photon evolution, so we derive it in advance here. Referring back to equation 2.27, the time component is given by:

\[
\Gamma_{ab} = \frac{\delta^{00}}{2} (g_{\alpha\beta} + g_{\alpha\mu} - g_{\alpha\beta,0}) \\
= -\frac{1}{2(1 + 2\Psi)} (g_{\alpha\beta} + g_{\alpha\mu} - g_{\alpha\beta,0}) \\
= -\frac{1}{2} (1 - 2\Psi) (g_{\alpha\beta} + g_{\alpha\mu} - g_{\alpha\beta,0})
\]

where in the last step I use the approximation \( 1/(1 + x) \approx 1 - x \) for \( x \ll 1 \). Continuing on, we calculate the elements of the time component, recalling that all terms of higher than linear order in \( \Psi \) and \( \Phi \) will be neglected:

\[
\Gamma_{ab} = -\frac{1}{2} (1 - 2\Psi) (g_{\alpha\beta,0} + g_{\alpha\mu} - g_{\alpha\beta,0}) \\
= -\frac{1}{2} (1 - 2\Psi) \frac{\partial}{\partial x^a} (g_{00}) \\
= \frac{1}{2} (1 - 2\Psi) \frac{\partial}{\partial x^a} (1 + 2\Psi) \\
= (1 - 2\Psi) \frac{\partial}{\partial x^a} \\
\Gamma_{ab} = \Gamma_{0a} = \Psi, a
\]

\[
\Gamma_{ij} = -\frac{1}{2} (1 - 2\Psi) (g_{\alpha\beta} + g_{\alpha\mu} - g_{\alpha\beta,0}) \\
= \frac{1}{2} (1 - 2\Psi) (g_{00}) \\
= \delta_i^j \frac{1}{2} (1 - 2\Psi) \frac{\partial}{\partial t} [a^2(1 - 2\Phi)]
\]
\[
\begin{align*}
&= \delta_{ij} (1 - 2\Psi) \left[ a (1 - 2\Phi) \frac{da}{dt} - a^2 \frac{\partial \Phi}{\partial t} \right] \\
&= \delta_{ij} \left[ \frac{da}{dt} - a^2 \frac{\partial \Phi}{\partial t} - 2a (\Phi + \Psi) \frac{da}{dt} \right] \\
\Gamma_{ij}^0 &= \delta_{ij} a^2 \left[ H - \Phi, 0 - 2H (\Phi + \Psi) \right]
\end{align*}
\]

3.2 THE BOLTZMANN EQUATION FOR PHOTONS

The Boltzmann equation determines the evolution of the distribution function, \( f \), of a species.\(^2\) In this section, we seek to apply it to our perturbed universe so that we can begin quantifying the evolution of CMB temperature anisotropies.

3.2.1 COLLISIONLESS FORM FOR AN ARBITRARY DISTRIBUTION

In its collisionless form, the Boltzmann equation is simply:

\[
\frac{df}{dt} = 0 \tag{3.5}
\]

In our previous discussion of the photon distribution function (§2.1), we allowed the distribution function to be a function of photon momentum (energy) only. In our perturbed universe, we must allow more freedom. In general, the distribution function of a species can vary as a function of position, momentum, and time: \( f = f(t, x^i, P^\mu) \). We can consolidate the momentum dependence by noting that the contraction of the four-momentum is invariant:

\[
P^2 = P_i P^i = g_{\mu\nu} P^\mu P^\nu = m = 0 \tag{3.6}
\]

where the last equality clearly is only true of massless particles (such as the photons we are dealing with). Identifying the spatial component \( g_{ij} P_i P_j \) of the contraction with the squared magnitude of the three-momentum, \( p^2 \), we express equation 3.6 as follows:

\[
p^2 = -g_{00} P^0 = (1 + 2\Psi)(P^0)^2 \tag{3.7}
\]

\[
P^0 = E = \frac{p}{\sqrt{1 + 2\Psi}} = p(1 - \Psi)
\]

where we have used the approximation \( \sqrt{1 - x} = 1/(1 - x/2) \) for \( x \ll 1 \).

We can now re-express the four-momentum in terms of the magnitude of the three-momentum, \( p \), and the direction of photon propagation \( \hat{p}^i \), which is a unit three-vector. This allows us to express the distribution function as \( f = f(t, x^i, p, \hat{p}^i) \). Expanding out the derivative in equation 3.5, we have:

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial p} \frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i} \frac{d\hat{p}^i}{dt} \tag{3.8}
\]

\(^2\)This section heavily relies on Dodelson (2003), §4, and to a lesser extent, on Ma and Bertschinger (1995) §5.
We will have to clean up these terms to make them manageable — note that the first term is already fine.

The second term can be re-expressed using the affine parameter (equation 2.29):

\[
\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{P^i}{E} = \frac{P^i}{p(1 - \Psi)}
\]

It is tempting to claim that \( \frac{P^i}{p} = \hat{p}^i \), but this is decidedly not true in a non-Euclidean metric. We must instead allow the relationship to be defined by a more general proportionality factor: \( P^i = C\hat{p}^i \). We can determine this parameter via our definition of \( p \):

\[
p^2 = g_{ij}P^iP^j = C^2g_{ij}\hat{p}^i\hat{p}^j = C^2a^2(1 - 2\Phi)\delta_{ij}\hat{p}^i\hat{p}^j
\]

where the \( \delta_{ij} \) term comes from the fact that the metric is diagonal. Since \( \hat{p}^i \) is a unit vector by definition, \( \delta_{ij}\hat{p}^i\hat{p}^j = (\hat{p}^1)^2 + (\hat{p}^2)^2 + (\hat{p}^3)^2 = 1 \). Therefore:

\[
C = \frac{p}{a\sqrt{1 - 2\Phi}} = \frac{p(1 + \Phi)}{a}
\]

\[
\frac{dx^i}{dt} = \frac{C\hat{p}^i}{p(1 - \Psi)} = \frac{\hat{p}^i(1 + \Phi)}{a(1 - \Psi)} = \frac{\hat{p}^i}{a} (1 + \Phi)(1 + \Psi) = \frac{\hat{p}^i}{a} (1 + \Phi + \Psi)
\]

Considering the term \( \frac{\partial f}{\partial x^i} \), recall that our zero-order photon distribution function (the Bose-Einstein distribution, equation 2.3) has no position \( (x^i) \) dependence. Hence, \( \frac{\partial f}{\partial x^i} \) is only nonzero in first-order perturbations. This allows us to neglect the term in \( (\Phi + \Psi)\frac{\partial f}{\partial x^i} \), and we are left with the following form for the second term of equation 3.8:

\[
\frac{\partial f}{\partial x^i} \frac{dx^i}{dt} = \frac{\hat{p}^i}{a} \frac{\partial f}{\partial x^i}
\]

Moving along to the third term of equation 3.8, we recall the time component of the geodesic equation (2.31):

\[
P^0 \frac{dP^0}{dt} = -\Gamma^0_{\alpha\beta} P^\alpha P^\beta
\]

\[
p(1 - \Psi) \frac{d}{dt} [p(1 - \Psi)] = -\Gamma^0_{\alpha\beta} P^\alpha P^\beta
\]

\[
(1 - \Psi) \left[ \frac{dp}{dt} - \mathbf{d} \frac{\Psi}{dt} - \Psi \frac{dp}{dt} \right] = -\Gamma^0_{\alpha\beta} P^\alpha P^\beta
\]

\[
\frac{dp}{dt} (1 - 2\Psi) - p \frac{d\Psi}{dt} = -\Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{p}
\]

\[
\frac{dp}{dt} (1 - 2\Psi) = p \left( \frac{d\Psi}{dt} + \frac{\partial \Psi}{\partial x^i} \frac{dx^i}{dt} \right) - \Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{p}
\]

\[
\frac{dp}{dt} = p \left( \Psi, 0 + \hat{p}^i a \Psi, i \right) - \Gamma^0_{\alpha\beta} \frac{P^\alpha P^\beta}{p} (1 + 2\Psi)
\]

where in the last statement, we multiplied both sides by \( (1 + 2\Psi) \) and dropped all second-order
terms. Consider the Christoffel symbol term by itself for a moment:

$$\frac{1}{p} \Gamma^a_{\alpha\beta} P^\alpha P^\beta = \frac{1}{p} \left[ \Gamma^0_{\alpha\beta} P^\alpha P^0 + 2 \Gamma^0_{\alpha\beta} P^0 P^\beta + \Gamma^\alpha_{\alpha\beta} P^\beta \right]$$  \hspace{1cm} (3.14)

Recalling that \( P^0 = p(1 - \Psi) \) and \( P^\beta = p\hat{\beta}(1 + \Phi)/a \), and dropping all second order terms, we have:

$$\frac{1}{p} \Gamma^a_{\alpha\beta} P^\alpha P^\beta = \frac{1}{p} \left[ p^2 \Psi,0 + \frac{2p^2}{a} \hat{\beta} \Psi,0 + \delta_{ij} P^i P^j (H - \Phi,0 - 2H(\Phi + \Psi)) \right]$$  \hspace{1cm} (3.15)

Care must be taken with the \( \delta_{ij} P^i P^j \) term. Note that \( \delta_{ij} P^i P^j = \delta_{ij} g^{ij} P^i P^j = p^2 (1 + 2\Phi)/a^2 \). Substituting this in, we have:

$$\frac{1}{p} \Gamma^a_{\alpha\beta} P^\alpha P^\beta = p \Psi,0 + \frac{2p\hat{\beta}}{a} \Psi,0 + p(1 + 2\Phi) (H - \Phi,0 - 2H(\Phi + \Psi))$$  \hspace{1cm} (3.16)

$$= p \Psi,0 + \frac{2p\hat{\beta}}{a} \Psi,0 + pH - p\Phi,0 - 2pH\Psi$$  \hspace{1cm} (3.17)

Returning to our progress with the geodesic equation (3.13) and inserting the above result, we arrive at:

$$\frac{dp}{dt} = p \left[ \Phi,0 - \frac{\hat{\beta}}{a} \Psi,0 - H \right]$$  \hspace{1cm} (3.18)

The final term in equation 3.8, \( \frac{\delta f}{\delta \hat{\beta}} \), is negligible because the distribution function only varies with photon direction to first order, and the photon direction does not change at zero order. Hence, this is the product of two first order terms, and can be neglected.

Combining all our results, we have finally arrived at the collisionless Boltzmann equation for photons:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{\beta}}{a} \frac{\partial f}{\partial x'} + p \frac{\partial f}{\partial \Phi} \left[ \frac{\partial \Phi,0}{a} - \frac{\hat{\beta}}{a} \frac{\partial \Psi,0}{a} - H \right] = 0$$  \hspace{1cm} (3.19)

### 3.2.2 Application to a Perturbed Distribution Function

To apply this to our universe, we need a perturbed form for the photon distribution function. We create this by allowing the photon temperature to deviate from its zero-order temperature (which is the same over all space). This temperature anisotropy is given by \( \Delta_T(t, x', \hat{\beta}') \), which is notably not a function of the magnitude of the momentum, \( p \).\(^3\) This temperature anisotropy function is inherently dimensionless, the temperature is given by \( T(t)(1 + \Delta_T) \), where \( T(t) \) is the zero-order average temperature of the photon field. Note that this \( \Delta_T \) gives the photon temperature anisotropy at all points in space \( (x') \) at all times \( (t) \) over the whole sky \( (\hat{\beta}') \). Hence, if we let \( x_0' \) denote our conformal location in the universe, then \( \Delta_T(t_0, x_0', \hat{\beta}') \) is equivalent to the sky map anisotropy function \( \delta T(\theta, \phi) \)

\(^3\)See Ma and Bertschinger (1995), §6 for justification. The justification rests on the fact that Compton scattering of photons off electrons in the early universe does not change the magnitude of the photon’s momentum, and hence the temperature anisotropy does not depend on momentum magnitude (Dodelson, 2003).
that we discussed in equation 1.16. We can now express our perturbed distribution function, which is formed simply by inserting our perturbed temperature into the Bose-Einstein distribution:

\[
f(t, x', p, \dot{p}') = \frac{1}{\exp \left( \frac{p}{T(1+\Delta_T)} \right) - 1}
\]  

(3.20)

To express this in a form that will be easier to insert into equation 3.19, we Taylor expand the unperturbed distribution function \( f^0 = 1/(\exp \frac{p}{T} - 1) \) to first order:

\[
f(t, x', p, \dot{p}') = f^0 + T\Delta_T \frac{\partial f^0}{\partial T}
\]  

(3.21)

We now note a convenient identity relating derivatives of \( f^0 \):

\[
\frac{\partial f^0}{\partial T} = \frac{\partial}{\partial T} \left[ \frac{1}{\exp \frac{p}{T} - 1} \right] = \frac{p}{T^2} e^{p/T} [e^{p/T} - 1]^{-2} = -\frac{p}{T} \frac{\partial f^0}{\partial p}
\]  

(3.22)

This leaves us with our final form for the perturbed distribution function:

\[
f(t, x', p, \dot{p}') = f^0 - p\Delta_T \frac{\partial f^0}{\partial p}
\]  

(3.23)

Next, insert the distribution function of equation 3.23 into the final form for the Boltzmann equation, 3.19, recalling that \( f^0 \) is a function of \( p \) and \( t \) only:

\[
\frac{df}{dt} = \frac{\partial f^0}{\partial t} - p \frac{\partial}{\partial t} \left[ \Delta_T \frac{\partial f^0}{\partial p} \right] - p \frac{\partial}{\partial t} \left[ \frac{\partial f^0}{\partial x^i} \frac{\partial f^0}{\partial p} \right] + p \left[ \frac{\partial f^0}{\partial p} - \Delta_T \frac{\partial f^0}{\partial p} - p\Delta_T \frac{\partial f^0}{\partial p^2} \right] \left[ \frac{\partial \Phi}{\partial t} - \frac{\dot{p}}{a} \frac{\partial \Psi}{\partial x} - H \right]
\]  

(3.24)

\( \Delta_T \) is now one of our perturbation variables, so anything second order in any combination of \( \Delta_T, \Psi, \) and \( \Phi \) can be neglected:

\[
\frac{df}{dt} = \frac{\partial f^0}{\partial t} - pH \frac{\partial f^0}{\partial p} - p \frac{\partial}{\partial t} \left[ \Delta_T \frac{\partial f^0}{\partial p} \right] - p\Delta_T \frac{\partial}{\partial t} \frac{\partial f^0}{\partial p} - p \frac{\partial}{\partial t} \frac{\partial f^0}{\partial p^2} + p \left[ \frac{\partial f^0}{\partial p} + p \frac{\partial f^0}{\partial p^2} \right] + p \left[ \frac{\partial \Phi}{\partial t} - \frac{\dot{p}}{a} \frac{\partial \Psi}{\partial x} \right]
\]  

(3.25)

I now seek to show that the fourth term in the above expression is in fact equal to the sixth term. We begin by recasting \( \partial / \partial t \) as \( (dT / dt)(\partial / \partial T) \), and noting that we can exchange the order in which we differentiate \( f^0 \). This allows us to use the identity we established in equation 3.22:

\[
p\Delta_T \frac{\partial}{\partial t} \frac{\partial f^0}{\partial p} = p\Delta_T \frac{dT}{dt} \frac{\partial}{\partial T} \frac{\partial f^0}{\partial p} = p\Delta_T \frac{dT}{dt} \frac{\partial}{\partial p} \left[ -\frac{p}{T} \frac{\partial f^0}{\partial p} + \frac{\partial f^0}{\partial p} \right] = -p\Delta_T \frac{dT}{dt} \frac{\partial}{\partial p} \left[ \frac{\partial f^0}{\partial p} - p \frac{\partial f^0}{\partial p^2} \right]
\]  

(3.26)

---

4 One could counter this definition by saying that the Bose-Einstein distribution is actually \( 1/\left(\exp \frac{E}{T(1+\Delta_T)} - 1\right) \), and hence, the perturbed version should be \( \left[ \exp \frac{p(1+\Psi)}{T(1+\Delta_T)} \right]^{-1} \). However, we could change the \( (1 - \Psi) \) in the numerator to a \( (1 + \Psi) \) in the denominator, and then note that our \( \Delta_T \), as defined in the text, is simply \( \Delta_T^c + \Psi \). Our definition works better observationally, because when observing photons, we simply assume that \( E = p \), ignoring the gravitational potential energy of the photons.
At this point, all we need to do to reach the goal is establish that \( \frac{dT}{dt}/T = -H \). As we noted in §2.1.1, \( T \propto 1/a \), and hence, we have:

\[
\frac{dT}{dt}/T = \frac{\dot{a}}{a} - 1 = -H \tag{3.27}
\]

proving that the fourth term and the sixth term of equation 3.25 do in fact cancel. Note that this argument also implies that the first two terms cancel. This leaves us with the following form for \( df/dt \) in our perturbed universe:

\[
\frac{df}{dt} = -p \frac{\partial f^0}{\partial p} \left( \frac{\partial \Delta_T}{\partial t} + \frac{\dot{p} \cdot \Delta_T}{\partial x^i} - \frac{\partial \Phi}{\partial t} - \frac{\dot{p} \cdot \partial \Psi}{\partial x^i} \right) \tag{3.28}
\]

### 3.2.3 COLLISION TERMS

The full Boltzmann equation includes a term resulting from collisions with particles of other species (Ma and Bertschinger, 1995):

\[
\frac{df}{dt} = \left( \frac{df}{dt} \right)_C \tag{3.29}
\]

where the right-hand side of the equation is the collision term, which represents forced deviations caused by interactions with other species. For our CMB photons, the dominant interaction is Compton scattering off electrons. At early times (before recombination), this scattering was extremely efficient, and the mean free path of photons was quite short. A derivation of the photon-electron scattering term in our perturbed universe is given in Dodelson (2003), §4.3, and in Kosowsky (1996). I cite the final result here:

\[
\left( \frac{df}{dt} \right)_C = -p \frac{\partial f^0}{\partial p} n_e \chi_e \sigma_T \left[ \Delta_{T_0} - \Delta_T + i \dot{p} \cdot \vec{v}_b \right] \tag{3.30}
\]

This statement introduces some new notation that I explain in the following paragraphs.

The electrons are described by their density, \( n_e \), ionization fraction \( \chi_e \), and bulk velocity, \( \vec{v}_b \). The ionization fraction, \( \chi_e \), is normalized to the number of Hydrogen atoms (protons) in the universe. Hence, “full” (\( \chi_e = 1 \)) ionization means that all Hydrogen atoms are ionized. It is possible for \( \chi_e \) to be greater than one — this intuitively means that some Helium is ionized in addition to most (if not all) of the Hydrogen. Noting that \( \sigma_T \) is the Thomson cross-section of the electron-proton interaction, we can identify the quantity \( n_e \chi_e \sigma_T \) as the linear density of electrons along a line of sight. This leads us to define the optical depth, which counts the number of electrons a photon

---

5 One could argue that I have done this backwards. Dodelson (2003) sets the first two terms (the zero order terms) to zero and derives \( T \propto 1/a \). I choose to assume the correctness of the hand-waving derivation of §2.1.1.

6 Although my derivations closely follow Dodelson (2003), my notation follows that of Seljak and Zaldarriaga (1996). Equation 3.30 introduces a new required translation between the two notations. Namely, Dodelson’s electron bulk velocity, \( \vec{v}_b \) \( \text{Dodelson} \) is equivalent to \( i \vec{v}_b \), where \( \vec{v}_b \) is the bulk velocity used in this paper and in Seljak and Zaldarriaga. This notation is used so that the final form of the evolution equation for \( \Delta_T \) can be treated as a real-valued equation, making it computationally simpler.

7 In fact, \( \chi_e \) is greater than one at extremely early times, when the universe is so hot that not even Helium nuclei can capture electrons.
interacts along a line of sight:

\[ \kappa(t) = \int_{t_0}^{t} n_e \chi e \sigma_T dt' \]
\[ \kappa(\tau) = \int_{\tau_0}^{\tau} n_e \chi e \sigma_T a d\tau' \]

where on the second line, I have translated to conformal time (in preparation for translating our entire progress thus far to conformal time). Intuitively, \( \kappa(t) \) is the average number of electrons a photon arriving at our detectors today has interacted with since time \( t \). We expect \( \kappa \) to be very large at times before recombination, and significantly less than one for times after recombination (this has to be the case, otherwise we cannot honestly speak of a well-defined “surface of last scattering”). Note, however, that \( \kappa \) is not zero for all times after recombination, due to the reionization of the universe which has occurred at recent times.\(^8\) The optical depth to the surface of last scattering is the standard parameter used to describe the reionization effect. WMAP results indicate that \( \kappa(t_{\text{rec}}) = \kappa_{\text{rec}} \approx 0.09 \), which corresponds to semi-abrupt full reionization that occurred at redshift \( z \approx 11 \) (Spergel et al., 2006).

Returning to equation 3.30, we still need to explain the term \( \Delta T_0 \). This term depends only on \( x^i \) and \( t \), while \( \Delta T \) also depends on \( p^i \). We form \( \Delta T_0 \) from \( \Delta T \) by integrating over all possible photon directions, \( p^i \):

\[ \Delta T_0(t, x^i) = \frac{1}{4\pi} \int \Delta T(t, x^i, p^i) d\Omega \]

Hence, \( \Delta T_0 \) is the average photon temperature anisotropy at a given point in space and time. We will generalize this to arbitrary multipoles shortly.

### 3.2.4 Putting it All Together

Now that we have come to grips with the components of the collision term, we can insert it into the general Boltzmann equation (3.29). Recalling our result from §3.2.2 for \( df/dt \), we have:

\[ \frac{\partial \Delta T}{\partial t} + \frac{p^i}{a} \frac{\partial \Delta T}{\partial x^i} - \frac{\partial \Phi}{a} + \frac{p^i}{a} \frac{\partial \Psi}{\partial x^i} = n_e \chi e \sigma_T \left[ \Delta T_0 - \Delta T + i p \cdot \vec{v}_b \right] \]

First, we switch to conformal time, via the identity \( dt = a d\tau \). Multiplying through by a factor of \( a \), we have:

\[ \frac{\partial \Delta T}{\partial \tau} + \frac{p^i}{a} \frac{\partial \Delta T}{\partial x^i} - \frac{\partial \Phi}{\partial \tau} + p^i \frac{\partial \Psi}{\partial x^i} = n_e \chi e \sigma_T a \left[ \Delta T_0 - \Delta T + i \hat{p} \cdot \vec{v}_b \right] \]

Now, we are in a position where we can take the Fourier transform of this entire equation. As we discussed in §1.5, this will allow us to consider our perturbations as a function of their scale, not their absolute position in space. We are transforming the position variable \( x^i \), so derivatives with respect to \( x^i \) transform to \( ik_i \) (\( \frac{\partial}{\partial x^i} \rightarrow ik_i \)). Note that \( k_i = k' \), since our \( k \)-space is Euclidean. After

---

\(^8\)This reionization occurs because neutral hydrogen in gas clouds becomes ionized due to excitation by stellar radiation.
transforming equation 3.34, we are left with:

\[
\dot{\Delta}_T + i\hat{p}^{i} k_{i} \Delta_T - \Phi + i\hat{p}^{i} k_{i} \Psi = n_{e} \chi_{e} a \left[ \Delta_{\Theta} - \Delta_T + i\hat{p} \cdot \hat{k} v_{k} \right]
\]  

(3.35)

where, from now on, overdots denote derivatives with respect to conformal time, and we do not explicitly denote that we have transformed the perturbation parameters — when we see them, we should interpret them as Fourier-space functions. Also, in equation 3.35, we have assumed that the bulk velocity of the electrons is in the \( \hat{k} \) direction, so the Fourier transform of \( \hat{p} \cdot \hat{v} \) is \( \hat{p} \cdot \hat{k} v_{k} \), where \( v_{k} \) is the Fourier transform of the scalar velocity.

To further simplify things, we define \( \mu \) to be the angle between the photon propagation direction and the wavevector, so \( \mu = \hat{p} \cdot \hat{k} \). Next, we assume that \( \Delta_{T}(\tau, k', \hat{p}') \) depends on the photon propagation direction \( \hat{p}' \) only through this angle \( \mu \). This is equivalent to claiming that \( \Delta_{T} \) does not depend on the phase of the photon direction. Now, \( \Delta_{T} \) is a function of \( \tau, k, \) and \( \mu \). This leaves the following final form for the Boltzmann evolution equation:

\[
\dot{\Delta}_T + i k \mu \Delta_T = \Phi - i k \mu \Psi - \dot{\kappa} \left[ \Delta_{\Theta} - \Delta_T + i \mu v_{k} \right]
\]  

(3.36)

where I have identified the quantity \( n_{e} \chi_{e} \sigma_{T} a \) as \( -\dot{\kappa} \), using the definition of equation 3.31.\(^9\)

Now, we should note that equation 3.36 has no explicit dependence on the direction of the wave vector. Hence, our perturbation parameters evolve independently of \( \hat{k} \). All the phase information is therefore contained in the initial conditions, which are set up by inflation (so by “initial”, we are really speaking of the conditions immediately following inflation).\(^10\) This allows us to re-define the parameters in equation 3.36 to be independent of \( \hat{k} \) by dividing through by the initial conditions of one of the parameters. We choose to use the initial perturbation to the potential, \( \Psi_{i}(k') \). For example, we can write \( \Delta_{T}(\tau, k', \mu) \) as follows:

\[
\Delta_{T}(\tau, k', \mu) = \Psi_{i}(k') \Delta_{T}(\tau, k, \mu)
\]  

(3.37)

If we make the similar substitution for every perturbation parameter in equation 3.36, the initial conditions simply cancel from every term, and the equation is unchanged, except for how we interpret the parameters. After the substitution, \( \dot{\kappa} \) is still a function of \( \tau \) only; \( \Phi, \Psi, \) and \( v_{k} \) are functions of \( \tau \) and \( k \); and \( \Delta_{T} \) is a function of \( \tau, k, \) and \( \mu \) (as in the right-hand side of equation 3.37).

\(^9\)In this instance, I follow the notation of Dodelson (2003), rather than that of Seljak and Zaldarriaga (1996). In the latter work, \( n_{e} \chi_{e} \sigma_{T} a \) is defined to be \( \dot{\kappa} \) and the optical depth is defined to be \( \int_{0}^{\tau} \dot{\kappa} d\tau \). These definitions are not mathematically consistent, so I will stick with Dodelson’s notation. Any appearance of \( \dot{\kappa} \) in this paper can be translated to Seljak and Zaldarriaga’s notation via a sign flip.

\(^10\)I am intentionally skipping over the involved theory of how inflation sets up these initial conditions. The leading theory is that some scalar potential field drives inflation and is responsible for setting up the initial perturbations to our parameters. In our study, we will only consider adiabatic initial conditions, in which the ratio of matter to radiation is the same everywhere (2003).
It is more convenient to express the $\mu$ dependence of $\Delta T$ by taking multipole moments:\footnote{The convenience of the multipole decomposition is due to the fact that, using the $\Delta_T$ form, we can very easily cut off our calculations at a given angular scale, by only sampling out to some maximum $\ell$. In addition, we will see in the next chapter that this multipole decomposition relates directly to the spherical harmonic decomposition of the CMB temperature map, which we introduced in §1.17.}

\[
\Delta_T(k, \tau) = \frac{1}{2(-i)^\ell} \int_{-1}^{1} P_\ell(\mu) \Delta_T(\mu, k, \tau) d\mu
\]  

(3.38)

where $P_\ell$ is the Legendre polynomial of order $\ell$. It is also useful at this juncture to note how to invert this expansion. Using the orthogonality property of the Legendre polynomials:

\[
\int_{-1}^{1} P_\ell(\mu) P_\ell'(\mu) d\mu = \frac{1}{2\ell + 1} \delta_{\ell\ell'}
\]  

(3.39)

we can invert equation 3.38 as follows:

\[
\Delta_T(\mu, k, \tau) = \sum_{\ell} (-i)^\ell (2\ell + 1) P_\ell(\mu) \Delta_T(\ell, k, \tau)
\]  

(3.40)

We have not, as of yet, included or discussed effects due to photon polarization. The polarization of the photons does enter into the collision term of the Boltzmann equation. If we had included polarization, we would have the following version of equation 3.36:

\[
\dot{\Delta}_T + i k \mu \Delta_T = \Phi - i k \mu \Psi - \kappa \left[ \Delta_\text{P}_0 - \Delta_T + i \mu v_b - \frac{1}{2} P_2(\mu) \Pi \right]
\]  

(3.41)

Where the $\Pi$ term encapsulates the polarization information:

\[
\Pi = \Delta_{\text{P}_1} + \Delta_{\text{P}_2} + \Delta_\text{P}_0
\]  

(3.42)

Where $\Delta_\text{P}(\tau, k, \mu)$ is the polarization anisotropy. We will generally ignore polarization terms in our derivations, but we should remember that a full treatment does include them, and the programs that calculate the CMB anisotropy spectrum account for them.

### 3.3 Additional Equations

Equation 3.36 is a single equation describing the evolution of four perturbation parameters ($\Delta_T$, $\Phi$, $\Psi$, and $v_b$), and equation 3.41 requires us to solve for $\Delta_\text{P}$ as well. Clearly, we will need some more equations if we are going to successfully evolve these parameters. In this section, we simply cite and explain the additional equations that are necessary.\footnote{The results I cite are taken from Dodelson (2003) and Seljak and Zaldarriaga (1996). As has been my custom, I will use the notation of Seljak and Zaldarriaga.}

We have already had a brief encounter with the photon polarization anisotropy, $\Delta_\text{P}(\tau, k, \mu)$. The derivation of the Boltzmann equation governing its evolution follows along similar lines as the
derivation we performed for the photon temperature. The end result is:

\[
\Delta_P + ik\mu\Delta_P = -\dot{\kappa}\left[-\Delta_P + \frac{1}{2}(1 - P_c(\mu))\Pi\right]
\] (3.43)

This gives us two equations and five parameters.

Next, we consider the evolution of the dark matter. We will only consider cold (nonrelativistic) dark matter, and we denote its fractional overdensity by \(\delta_c\) and its bulk velocity by \(v_c\). The Boltzmann treatment can be applied to matter as well, giving us the following two equations:

\[
\dot{\delta}_c + kv_c = 3\dot{\Phi}
\] (3.44)

\[
\dot{v}_c + \frac{\dot{a}}{a}v_c = k\Psi
\]

We now have four equations and seven parameters.

Baryonic matter evolves in a similar fashion to the dark matter. We have already met the bulk velocity of baryons, \(v_b\), and we now define their fractional overdensity as \(\delta_b\). The equations governing the baryons are similar in form to those governing the dark matter:

\[
\dot{\delta}_b + kv_b = 3\dot{\Phi}
\] (3.45)

\[
\dot{v}_b + \frac{\dot{a}}{a}v_b = c_s^2 k\delta_b + \frac{4\rho_r}{3\rho_b}k(v_b - 3\Delta_T) + k\Psi
\]

Where \(c_s\) is the speed of sound in the baryon fluid and \(\rho_r, \rho_b\) are the photon (not radiation, which includes neutrinos) and baryon densities, respectively. All three of these can be calculated in-place, so we do not consider them to be new parameters. Hence, we now have six equations and eight parameters. The baryons are coupled to the photons here via the dipole of the photon distribution, \(\Delta_T\).

Neutrinos evolve in a similar manner to photons, although their evolution is less complicated since they are not coupled to the matter. We will denote the fractional temperature anisotropy of neutrinos as \(N\):

\[
\dot{N} + ik\mu N = \Phi - ik\mu\Psi
\] (3.46)

We now stand at seven equations and nine parameters.

The final two equations come from the time-time and time-space components of the Einstein
equation (Ma and Bertschinger, 1995):

\[ k^2 \Phi + 3 \frac{\dot{a}}{a} \left( \Phi + \frac{\dot{\Psi}}{a} \right) = -4\pi G a^2 \delta \rho \]  \hspace{1cm} (3.47)

\[ k^2 \left( \Phi + \frac{\dot{\Psi}}{a} \right) = 4\pi G a^2 \delta f \]

Here, \( \delta \rho \) denotes the total density perturbation (accounting for all species), and \( \delta f \) is the total momentum density perturbation (for all species). We can express these in terms of the parameters we have already specified as follows:

\[ \delta \rho = \rho_i \delta_i + \rho_b \delta_b + 4 \rho_\gamma \Delta T_0 + 4 \rho_\nu N_0 \]  \hspace{1cm} (3.48)

\[ \delta f = (\rho_b + P_b) v_b + (\rho_c + P_c) v_c \]

where \( P \) denotes the pressure of the species. The factors of four for the radiation components in the first equation can be explained by noting that \( \rho_\gamma \propto T^4 \), so \( \rho_\gamma \propto (\langle T \rangle (1 + \Delta T_0))^4 = \langle T \rangle^4 (1 + 4\Delta T_0) \).

Hence, we can identify \( \delta \rho \) as \( 4\Delta T_0 \). We now have all the required equations and parameters we need to calculate the evolution of the universe.
Now that we have all the equations necessary to describe the evolution of the universe (or, at least, our perturbation parameters), we can go about solving them. We would like to solve for the multipole moments of the temperature anisotropy, \( \Delta T_\ell(\tau_0, k) \), which in turn are used to derive today’s temperature angular power spectrum, \( C_\ell \). Traditionally, the \( \Delta T_\ell \) s were computed simultaneously via the following coupled system of differential equations:

\[
\begin{align*}
\dot{\Delta}_{\ell = 0} &= -k\Delta_{\ell = 1} + \Phi \\
\dot{\Delta}_{\ell = 1} &= \frac{k}{3}(\Delta_{\ell = 0} - 2\Delta_{\ell = 2} + \Psi) - \dot{\kappa} \left( \frac{\Pi}{3} - \Delta_{\ell = 1} \right) \\
\dot{\Delta}_{\ell = 2} &= \frac{k}{5}(2\Delta_{\ell = 1} - 3\Delta_{\ell = 3}) - \dot{\kappa} \left( \frac{10}{11} - \Delta_{\ell = 2} \right) \\
\dot{\Delta}_{\ell > 2} &= \frac{k}{2\ell + 1}(\ell\Delta_{(\ell-1)} - (\ell + 1)\Delta_{(\ell+1)}) - \dot{\kappa}\Delta_{\ell}, \quad \ell > 2
\end{align*}
\]

If we are interested in solving for the CMB spectrum out to multipoles of order 1000, this method requires us to evolve a coupled system of several thousand differential equations. While tractable, this method requires a large amount of computational time. Another approach to calculating the \( \Delta T_\ell \) s, introduced by Seljak and Zaldarriaga (1996), and implemented in their CMBFAST\(^2\) program, reduces the number of equations, and thus the run time, dramatically. The calculations performed in this paper were made using the CAMB\(^3\) code base (Lewis et al., 2000), which is a generalization of the CMBFAST code.

In this chapter, we explain the line of sight integration approach to CMB anisotropies of Seljak and Zaldarriaga (1996), and describe the final calculation that allows us to predict the power spectrum, \( C_\ell \). We will also discuss some interesting features of the CMB power spectrum.

### 4.1 The Line of Sight Integration Approach

The line of sight integration scheme begins with equation 3.36, which we derived in the last chapter.\(^4\) Begin by reorganizing the equation slightly (subtract \( \dot{\kappa}\Delta_\ell \) from both sides).

\[
\dot{\Delta}_\ell + (ik\mu - \dot{\kappa})\Delta_\ell = \Phi - ik\mu\Psi - \dot{\kappa} \left[ \Delta_{\ell = 0} + i\mu v_\mu \right]
\]
Next, we note that the left hand side can be re-written as a time derivative:

$$e^{-ik\mu \tau} \frac{d}{d\tau} [\Delta_T e^{ik\mu \tau}] = \Phi - ik\mu \Psi - \kappa [\Delta_{\tau_0} + i\mu \nu_b]$$

(4.3)

Now, move the leading exponential to the right-hand side, and integrate over time. We will keep the limits of this integration arbitrary for now — use $\tau_i < \tau_f$ to denote initial and final times, respectively:

$$\int_{\tau_i}^{\tau_f} \frac{d}{d\tau} [\Delta_T e^{ik\mu \tau}] d\tau = \int_{\tau_i}^{\tau_f} e^{ik\mu \tau} [\Phi - ik\mu \Psi - \kappa \Delta_{\tau_0} - i\kappa \mu \nu_b] d\tau$$

(4.4)

The integral on the left-hand side is trivial:

$$\Delta_T(\tau_i, k, \mu) e^{ik\mu \tau_f - \kappa} - \Delta_T(\tau_i, k, \mu) e^{ik\mu \tau_i - \kappa} = \int_{\tau_i}^{\tau_f} e^{ik\mu \tau - \kappa} [\Phi - ik\mu \Psi - \kappa \Delta_{\tau_0} - i\kappa \mu \nu_b] d\tau$$

(4.5)

where we have introduced the notation that $\kappa_i = \kappa(\tau_i)$ and $\kappa_f = \kappa(\tau_f)$ denote the optical depth to the initial and final times, respectively. We now assume that $\tau_f$ is some very early time — well before recombination. In this case, we expect $\kappa_i$ to be extremely large, and we can neglect the term that contains $e^{-\kappa_i}$. Hence:

$$\Delta_T(\tau_f, k, \mu) = \int_{\tau_i}^{\tau_f} e^{i\kappa_f \mu \tau_f} e^{ik\mu \tau_f - \kappa_f \mu \tau_f} d\tau$$

(4.6)

Now, our goal is to integrate over $\mu$ so that we can separately consider each multipole moment of $\Delta_T$. We cannot do this unless we find a way to consolidate the $\mu$ dependence in the integrand into the exponential term. Note that whenever $\mu$ appears in the bracketed term in the integrand, it multiplies some function of $k$ and $\tau$ (such as $\Psi$). So consider an arbitrary term $\mu F(k, \tau)$ in the bracketed term individually:

$$\int_{\tau_i}^{\tau_f} \mu F(e^{ik\mu \tau}) e^{i\kappa \mu \tau} d\tau = \frac{1}{i\kappa} \int_{\tau_i}^{\tau_f} F(e^{ik\mu \tau}) \frac{d}{d\tau} [e^{ik\mu \tau}] d\tau$$

(4.7)

This expression can be integrated by parts, with $u = F e^{ik\mu \tau}$ and $dv = \frac{2}{\kappa} [e^{ik\mu \tau}] d\tau$.

$$\int_{\tau_i}^{\tau_f} \mu F(e^{ik\mu \tau}) e^{i\kappa \mu \tau} d\tau = \frac{1}{i\kappa} \left[ F e^{i\kappa \mu \tau} - F e^{-i\kappa \mu \tau} e^{ik\mu \tau} \right]_{\tau_i}^{\tau_f} + \int_{\tau_i}^{\tau_f} \frac{d}{d\tau} [F e^{ik\mu \tau}] d\tau$$

(4.8)

We can drop the second boundary term because it contains $e^{-\kappa_i}$, and we note that the first boundary term, while non-negligible, has no $\mu$ dependence, so it will only contribute to the monopole moment of the temperature anisotropy. We have no way to distinguish a monopole moment in the CMB we observe today from the baseline mean temperature, so we say that the monopole moment is “unobservable”, and hence there is no reason to worry about the first boundary term. Our mathematical manipulations have succeeded, providing us with a “formula” for removing the $\mu$

---

5This statement also holds true for the form that includes polarization effects — where $P_2(\mu)$ multiplies $\Pi$. 
dependence from the bracketed term in the integrand of equation 4.6:

\[
\int_{\tau_i}^{\tau_f} \mu Fe^{ik(\tau - \tau_f)} e^{\gamma e^{-x}} d\tau = -\frac{1}{ik} \int_{\tau_i}^{\tau_f} e^{ik(\tau - \tau_f)} \frac{\partial}{\partial \tau} [Fe^{\gamma e^{-x}}] d\tau
\]  

(4.9)

Applying this formula, we arrive at this form for \( \Delta_T \):

\[
\Delta_T(\tau_f, k, \mu) = \int_{\tau_i}^{\tau_f} e^{ik(\tau - \tau_f)} S(\tau, k) d\tau
\]  

(4.10)

where we have defined the source function, \( S(\kappa, \tau) \), which contains all the physics of the evolution:

\[
S(\kappa, \tau) = e^{\kappa e^{-x}} \left[ \Phi - \kappa \Delta T_0 \right] + \frac{\partial}{\partial \tau} \left[ e^{\kappa e^{-x}} \left( \Psi - \frac{\kappa}{k} \nu_b \right) \right]
\]  

(4.11)

We should pause to note how our source function differs from those that are defined in the standard derivations ((Dodelson, 2003), (Seljak and Zaldarriaga, 1996)). In these derivations, \( \tau_f \) is simply set to \( \tau_0 \), under the assumption that we are only interested in the present time. Since, by definition, the optical depth at the present time, \( \kappa_0 = \kappa(\tau_0) \), is zero, this affects the exponential terms in the source function, \( e^{\kappa e^{-x}} \), simply reducing them to \( e^{-x} \). However, this paper aims to generalize the CMB calculation to arbitrary times, so we cannot make this assignment of \( \tau_f \), and have therefore left it arbitrary, leaving us with a more general version of the source function.

Now that we have consolidated the \( \mu \) dependence into the exponential term, we can transform to multipole moments:

\[
\Delta_{\ell}(\tau_f, k) = \int_{\tau_i}^{\tau_f} d\tau S(\tau, k) \left( \frac{1}{2(-i)\ell} \right) \int_{-1}^{1} d\mu P_{\ell}(\mu) e^{ik(\tau - \tau_f)}
\]  

(4.12)

We are now only one short mathematical substitution away from our final equation. Note the following property of the Legendre polynomials:

\[
\int_{-1}^{1} \frac{d\mu}{2} P_{\ell}(\mu) e^{ix\mu} = \frac{1}{(-i)^{\ell}} j_{\ell}(x)
\]  

(4.13)

where \( j_{\ell} \) is the spherical Bessel function of order \( \ell \). Applying this to our equation, and noting that \((-1)^{\ell} j_{\ell}(x) = j_{\ell}(-x)\), we have arrived at our final evolution equation:

\[
\Delta_{\ell}(\tau_f, k) = \int_{\tau_i}^{\tau_f} S(\tau, k) j_{\ell}[k(\tau_f - \tau)] d\tau
\]  

(4.14)

Before we press on, we should pause for a moment to consider the meaning of the source function. First, we note that we can reorganize equation 4.11 in the following way:

\[
S(\kappa, \tau) = e^{\gamma} \left[ -\kappa e^{-x} (\Delta T_0 + \Psi) + \frac{\partial}{\partial \tau} \left( -\kappa e^{-x} \frac{\nu_b}{k} \right) + e^{-x} (\Phi + \Psi) \right]
\]  

(4.15)

where I have moved the \( e^{\gamma} \) term outside. This is to highlight the fact that our more general deriv-
tion, with an arbitrary final time, only differs from the standard derivations by this factor.\footnote{Hence, our source function \( S \) is related to the source function of Seljak and Zaldarriaga (1996), \( S_{SZ} \), by the simple relationship \( S = e^{-\kappa}S_{SZ} \).} The simplicity of this relationship will make it easy to account for in our implementation, which we discuss in the next chapter.

Moving on, we can simplify equation 4.15 by defining the visibility function, \( g(\tau) = -\dot{\kappa} e^{-\kappa} \). Note that this function goes to zero at early times, since it is damped by the \( e^{-\kappa} \) term. In addition, it goes to zero at late times, due to the extremely small post-recombination scattering rate, \( \dot{\kappa} \). We should interpret \( g(\tau) \) as the probability that a photon last scattered at time \( \tau \) — it is strongly peaked around the time of recombination. Making this substitution, we have:

\[
S(k, \tau) = e^{\kappa f} \left[ g (\Delta T_0 + \Psi) + \frac{\partial}{\partial \tau} \left( g \frac{v_b(k)}{k} \right) + e^{-\kappa} (\Phi + \Psi) \right] \tag{4.16}
\]

The intuitive meaning of the terms in the source function is worth noting. The \( g(\Delta T_0 + \Psi) \) term is peaked around recombination due to the visibility function. This term encodes the inherent anisotropy that was present at the time of recombination. The term in \( v_b \) also makes a dominant contribution around the time of recombination; it represents coupling of the photons to the electrons. The third term makes an impact only after recombination (due to the \( e^{-\kappa} \) term) — it is known as the integrated Sachs-Wolfe term.

For completeness, we should note the source function that results from including the polarization terms. It is a fairly simple extension of the above equation (2003):

\[
S(k, \tau) = e^{\kappa f} \left[ g \left( \Delta T_0 + \Psi + \frac{\Pi}{4} \right) + \frac{\partial}{\partial \tau} \left( g \frac{v_b(k)}{k} \right) + e^{-\kappa} (\Phi + \Psi) + \frac{3}{4k^2} \frac{\partial^2}{\partial \tau^2} (g\Pi) \right] \tag{4.17}
\]

We should also note that the same treatment we have applied in this section can be applied to the Boltzmann equation for polarization anisotropy (equation 3.43). This gives us a similar integral formula for the polarization anisotropy at some time \( \tau_f \).

The beauty of the form in equation 4.12 is that it decomposes the calculation into two terms, a geometrical term (the spherical Bessel, \( j_\ell \)) that does not depend on the cosmological parameters, and a physical source term \( (S) \) that does not depend on the multipole moment. To perform the calculation, the differential equations 3.44-3.47 can be evolved to obtain the source function \( S(\tau) \) for a given \( k \). To calculate the low moments of the photon anisotropy (\( \Delta T_0 \) and the elements of \( \Pi \)), which contribute to the source function, we need to use the “old” hierarchy of differential equations (equations 4.1 along with their parallels for polarization and neutrinos), but since we only need the first few moments, we only need the first few equations of the hierarchy, and can safely cut off the contributions from higher moments (CMBFAST uses the first eight moments). This means that the line of sight approach reduces the number of simultaneous differential equations from the thousands to the dozens.
4.2 Calculating the Power Spectrum

The last section gave us the ability to compute $\Delta_T(\tau_f, k)$ at any final time $\tau_f$ we like. In this section, we describe how to convert this $\Delta_T(\tau_f, k)$ into a prediction for $C_\ell$ (introduced in §1.6), the power spectrum of the observed CMB. As we noted when we first introduced $\Delta_T$ in §3.2, the CMB sky map at some location $\vec{x}$ at time $\tau_f$ is simply $\Delta_T(\tau_f, \vec{x}, \hat{p})$. Generalizing the spherical harmonic decomposition of equation 1.17 then, we have:

$$\Delta_T(\tau_f, \vec{x}, \hat{p}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\tau_f, \vec{x}) Y_{\ell m}(\hat{p})$$  \hspace{1cm} (4.18)

recall also that the spherical harmonics obey the following orthogonality property:

$$\int Y_{\ell m}(\hat{p}) Y^*_{\ell m'}(\hat{p}) d\Omega = \delta_{\ell\ell'} \delta_{mm'}$$  \hspace{1cm} (4.19)

this property allows us to solve for the $a_{\ell m}$s in equation 4.18:

$$a_{\ell m}(\tau_f, \vec{x}) = \int Y_{\ell m}(\hat{p}) \Delta_T(\tau_f, \vec{x}, \hat{p}) d\Omega$$  \hspace{1cm} (4.20)

At this point, we should note that the quantity $\Delta_T(\tau_f, \vec{x}, \hat{p})$ is actually quite different from the $\Delta_T(\tau_f, k)$ that we can calculate via the techniques developed in the previous section. We must now undo our mathematical manipulations to relate the two versions of $\Delta_T$. The first step in the process is to invert the Fourier transform. Recalling our convention from equation 1.15, we have:

$$a_{\ell m}(\tau_f, \vec{x}) = \int d^3 k \left( e^{i \vec{k} \cdot \vec{x}} \right) \int d\Omega Y_{\ell m}^*(\hat{p}) \Delta_T(\tau_f, \vec{x}, \hat{p})$$  \hspace{1cm} (4.21)

To get the power spectrum, $C_\ell$, at $\tau_f$, we need to take the ensemble average of the $a_{\ell m}$s over all space:

$$C_\ell(\tau_f) = \left\langle a_{\ell m}(\tau_f, \vec{x}) a^*_m(\tau_f, \vec{x}') \right\rangle$$  \hspace{1cm} (4.22)

$$= \left\langle \int d^3 k \left( e^{i \vec{k} \cdot \vec{x}} \right) \int d^3 k' \left( e^{-i \vec{k}' \cdot \vec{x}} \right) \int d\Omega Y_{\ell m}^*(\hat{p}) \Delta_T(\tau_f, \vec{x}, \hat{p}) \int d\Omega' Y_{\ell m}(\hat{p}') \Delta_T^*(\tau_f, \vec{x}', \hat{p}') \right\rangle$$

The ensemble average drives the quantity $\vec{k} \cdot \vec{x} - \vec{k}' \cdot \vec{x}'$ to zero, and therefore the exponentials drop out of the calculation. Since the spherical harmonics are static, the ensemble average now only acts on the $\Delta_T$ terms:

$$C_\ell(\tau_f) = \int d^3 k \int d^3 k' \int d\Omega \int d\Omega' Y_{\ell m}^*(\hat{p}) Y_{\ell m}(\hat{p}') \left\langle \Delta_T(\tau_f, \vec{x}, \hat{p}) \Delta_T^*(\tau_f, \vec{x}', \hat{p}') \right\rangle$$  \hspace{1cm} (4.23)

Now, we recall the tricks we used to simplify $\Delta_T$ in the last chapter. First, recall that the angular dependence on $\hat{p}$ only enters through the angle it makes with $\vec{k}$: $\hat{p} \cdot \vec{k}$. Also, we can remove the dependence on $\vec{k}$ by factoring out the initial conditions. Combining these effects, we can make the following substitution:

$$\Delta_T(\tau_f, \vec{k}, \hat{p}) = \Psi_\ell(\vec{k}) \Delta_T(\tau_f, k, \hat{p} \cdot \vec{k})$$  \hspace{1cm} (4.24)
After making this substitution, our $\Delta T$ term no longer has any phase dependence, so it falls outside of the ensemble average. We are now left with the following expression for $C_\ell$:

$$C_\ell(\tau_f) = \int d^3k \int d^3k' \int d\Omega \int d\Omega' Y^*_\ell(\hat{p}) Y_{\ell m}(\hat{p}') \left< \Psi_i(\vec{k}) \Psi^*_i(\vec{k}') \right> \times \Delta_\tau(\tau_f, k, \hat{k} \cdot \hat{p}) \Delta_\tau(\tau_f, k', \hat{k}' \cdot \hat{p}') \quad (4.25)$$

The ensemble average term is now simply the power spectrum of the initial potential perturbation:

$$\left< \Psi_i(\vec{k}) \Psi^*_i(\vec{k}') \right> = \delta^3(\vec{k} - \vec{k}') P_\Psi(k) \quad (4.26)$$

where we implicitly assume that $P_\Psi$ refers to the initial power spectrum. This substitution simplifies our expression for $C_\ell$ significantly:

$$C_\ell(\tau_f) = \int d^3k P_\Psi(k) \int d\Omega \int d\Omega' Y^*_\ell(\hat{p}) Y_{\ell m}(\hat{p}') \Delta_\tau(\tau_f, k, \hat{k} \cdot \hat{p}) \Delta_\tau(\tau_f, k, \hat{k}' \cdot \hat{p}') \quad (4.27)$$

Next, we transform to multipole moments $\Delta_\tau$, by using equation 3.40:

$$C_\ell(\tau_f) = \int d^3k P_\Psi(k) \int d\Omega \int d\Omega' Y^*_\ell(\hat{p}) Y_{\ell m}(\hat{p}') \times \left[ \sum_{\ell'} (-i)^{\ell'} (2\ell' + 1) P_{\ell'}(\hat{k} \cdot \hat{p}) \Delta_{\ell'}(\tau_f, k) \right] \times \left[ \sum_{\ell''} (i)^{\ell''} (2\ell'' + 1) P_{\ell''}(\hat{k}' \cdot \hat{p}') \Delta_{\ell''}(\tau_f, k) \right] \quad (4.28)$$

We need to apply one more transformation that will make things even more complicated momentarily, but the payoff is near. Use the following property of the Legendre polynomials:

$$P_\ell(\hat{n} \cdot \hat{n}') = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{n}) Y^*_{\ell m}(\hat{n}') \quad (4.29)$$

This substitution, and a little reorganization, gives us the following:

$$C_\ell(\tau_f) = (4\pi)^2 \int d^3k P_\Psi(k) \times \left[ \sum_{\ell'} (-i)^{\ell'} \left( \sum_{\ell''=-\ell}^{\ell''} Y_{\ell'' m'}(\hat{k}) \int d\Omega Y^*_{\ell'' m'}(\hat{p}) Y_{\ell'' m''}(\hat{p}') \right) \Delta_{\ell''}(\tau_f, k) \right] \times \left[ \sum_{\ell''} (i)^{\ell''} \left( \sum_{\ell''=-\ell}^{\ell''} Y_{\ell'' m''}(\hat{k}) \int d\Omega Y_{\ell'' m''}(\hat{p}) Y^*_{\ell'' m''}(\hat{p}') \right) \Delta_{\ell''}(\tau_f, k) \right] \quad (4.30)$$

Note that I have moved the angular integrals (which operate over $\hat{p}$ and $\hat{p}'$) inside the summations, because all the angular dependence is now consolidated there. Noting the orthogonality property of the spherical harmonics (equation 4.19), the angular integrals drop out and require that $\ell', \ell'' = \ell$.
and \( m', m'' = m \), leaving:

\[
C_\ell(\tau_f) = (4\pi)^2 \int d^3k P_\Psi(k) Y_{\ell m}(\hat{k}) Y_{\ell m}^*(\hat{k}) \left[ \Delta_\ell(\tau_f, k) \Delta_\ell^*(\tau_f, k) \right]
\] (4.31)

Next, rewrite \( d^3k \) as \( k^2 d\Omega dk \), where \( d\Omega \) is taken to be the integral over the solid angle described by the \( \hat{k} \) vector, and \( k \) is the magnitude of the wavevector. This allows us to separately perform the angular integral over the spherical harmonics, which is unity. This leaves us with our final form for the power spectrum:

\[
C_\ell(\tau_f) = (4\pi)^2 \int k^2 P_\Psi(k) \left[ \Delta_\ell(\tau_f, k) \Delta_\ell^*(\tau_f, k) \right] dk
\] (4.32)

This allows us to convert the \( \Delta_\ell \)'s from equation 4.14 into a prediction of the observed CMB spectrum. As we will see in the next section, this prediction is sensitive to variations in the cosmological parameters, so comparing the observed CMB power spectrum to theoretical models is an excellent method for determining the parameters.

Now that we understand how the power spectrum relates to the \( \Delta_\ell \)'s, and how the \( \Delta_\ell \)'s evolve (via the source function), we should take a moment to consider some basic predictions for the form of the power spectrum. We know from our discussion in chapter 1 that we expect to see a series of acoustic peaks and troughs at regularly spaced intervals in \( \ell \), due to the acoustic oscillations of perturbations in the photon-baryon plasma. At angular scales larger than the horizon at recombination, acoustic oscillations did not operate (because the corresponding \( k \)-modes are too large to be in causal contact between inflation and recombination). Therefore, we expect to see the primordial perturbation spectrum at large angular scales. Using some approximations, it can be shown that the power spectrum at large scales (\( \ell < 30 \)) follows the simple relationship:

\[
C_\ell \propto \frac{1}{\ell(\ell+1)}
\] (4.33)

This relationship explains why the power spectrum is always plotted as \( \ell(\ell+1)C_\ell \) — we expect this quantity to be constant at large scales. This large-scale (low \( \ell \)) region of the power spectrum is called the Sachs-Wolfe plateau in honor of the people who first derived the relationship represented by equation 4.33 (Sachs and Wolfe, 1967). The integrated Sachs-Wolfe (ISW) term, which we introduced in equation 4.16, depends on the derivatives of the metric perturbations \( \Psi \) and \( \Phi \) after the epoch of recombination. It serves to enhance anisotropies during epochs when the dominant energy species changes, at scales on the order of the current horizon. The transfer from radiation to matter dominance occurs before recombination, but the tail end of the transition causes an early-time ISW effect just after recombination. This causes enhancement of the first acoustic peak, which is near the scale of the horizon at recombination. In addition, a late-time ISW effect occurs during the transition from matter to dark energy dominance (a period we are in the midst of now). We expect this effect to cause enhancement of the power spectrum at the largest angular scales in a \( \Lambda \) dominated universe.

Before we move on, we should recall the fact that both the \( \Delta_\ell \)'s and the \( C_\ell \)'s are inherently unitless, by our original definition of \( \Delta_T \) (equation 3.20). Theoretical power spectra are typically
plotted with observational temperature units, usually $\mu K^2$, so we will have to give our theoretical power spectrum units if we are to compare it with the observed power spectrum. The simplest way to do this is to multiply $C_\ell$ by the square of the average CMB temperature after the fact, $T_0^2/a^2(\tau_f)$, where $T_0$ is the average CMB temperature today. Note that in standard approaches, when $\tau_f$ is always set to $\tau_0$, we simply multiply by $T_0^2$. We need to keep the more general approach in mind throughout the next chapter, where we focus on generalizing the standard treatment to solving the problem of calculating CMB evolution.

4.3 **The Power Spectrum and the Cosmological Parameters**

Now that we have finished developing the standard theoretical model for calculating CMB anisotropies, we can pause for a moment to appreciate the power spectrum as an incredibly useful tool for gaining understanding of the way the universe works. By fitting the observed CMB power spectrum to a theoretical model, it is possible to specify the cosmological parameters to high precision. This precision is possible because the CMB power spectrum has many complex features that react in interesting ways as the different cosmological parameters vary. We explore this behavior of the power spectrum in this section.

Figures 4.1 and 4.2 show the CMB power spectrum for a variety of cosmological models, all of which are variations of the “Base Model”, which is the best-fit $\Lambda$CDM model from WMAP. Figure 4.1 shows the power spectrum plotted on a logarithmic scale, which accentuates the low-$\ell$ effects and the overall behavior of the spectrum, including the damping tail at high $\ell$. Figure 4.2 is on a linear scale, which accentuates the behavior of the power spectrum around the first few acoustic peaks.

It is clear from these plots that the power spectrum is quite sensitive to changes in the cosmological parameters. We will consider each “altered” parameter in turn, and try to gain some physical intuition for its effect on the power spectrum.

The expansion rate of the universe is set by the Hubble constant, $H_0$. Note that when this parameter is increased, the power spectrum shifts to larger angular scales. In a high-$H_0$ model, the expansion of the universe proceeds faster, and the universe is therefore younger than in the base model. This means that the surface of last scattering is closer to the observer than it is in the base model, and by our discussion in §1.7, we expect CMB features to be at larger angular scales. This effect is a precursor of what we will see in our more generalized model in which we allow the CMB to evolve to an arbitrary final time.

The next model is a matter dominated flat universe with no cosmological constant. This model also produces a younger universe than the base model, so the spectrum is shifted to larger angular scales. The most striking effect is the overall suppression of the power, especially in the area of the first acoustic peak. This is due to the fact that the early-time ISW effect is smaller in this model than it is in the base model. Since the matter-dominated model has a higher $\rho_m$, the epoch of matter-radiation equality occurs earlier than in the base model. This means that right after recombination,
Figure 4.1: The CMB temperature angular power spectrum for a variety of cosmological models on a logarithmic scale. The base model is the best-fit WMAP $\Lambda$CDM model.

Figure 4.2: The CMB temperature angular power spectrum for a variety of cosmological models on a linear scale. The base model is the best-fit WMAP $\Lambda$CDM model.
the transition to matter dominance is even more complete than in the base model, and thus the early-time ISW effect is much smaller.

As we discussed in §1.5, the pattern of peaks and troughs in the anisotropy spectrum is directly dependent on the sound speed in the photon-baryon plasma. In a model with greater baryon content, this sound speed will be smaller (think of the greater density of baryons causing the plasma to become more viscous). As we saw, the magnitude of the “fundamental” wavenumber is inversely proportional to this sound speed, so in a high-$\Omega_b$ model, we expect the fundamental wavenumber to be larger. This effect accounts for the increased spacing between the acoustic peaks that we see in the model with “increased $\Omega_b h^2$”.

We have not spent much time discussing curvature effects, as we have typically assumed a flat universe throughout. However, the “open model” shown in figures 4.1 and 4.2 demonstrates that the CMB anisotropy spectrum can help determine the curvature of the universe. The main effect apparent in the figures is that the spectrum for the open model is shifted to significantly smaller scales compared to the base model. We can gain some intuition for this effect with a simple two-dimensional geometrical argument. On a closed two dimensional surface, such as a sphere, rays converge (two rays will coincide at their point of origin and at the opposite pole of the sphere), while on an open surface (such as a hyperboloid), rays diverge. Figure 4.3 illustrates how this effect alters the paths of CMB photons. In a flat universe (figure 4.3a), the photons follow straight lines, while in the open universe (figure 4.3a), the photon paths diverge. This means that, in an open universe, features of a given scale on the surface of last scattering are projected to smaller angular scales on today’s CMB sky map than they are in the flat case.

![Diagram](https://example.com/diagram)

Figure 4.3: The paths of two CMB photons originating from the same point on the surface of last scattering in two different cosmological geometries. In an open universe, photon paths diverge (considering them in the reverse direction), causing features of the CMB manifest themselves at smaller angular scales than they do in the a flat universe.

To discuss the model with enhanced $\kappa_{\text{rec}}$ (the optical depth to the surface of last scattering), we must explore the effects of reionization on the CMB spectrum. The intuitive effect is that reionization causes photons to scatter off direct paths leading from the surface of last scattering to us.

---

10 In this model, we maintain a flat universe, but set $\Omega_b h^2$ to twice its base model value.

11 The “open model” is identical to the base model, except that we remove the cosmological constant (effectively “replacing” $\Omega_\Lambda$ with $\Omega_K$).

12 In this model, we hold all other parameters constant and change $\kappa_{\text{rec}}$ to three times its base model value.
However, due to the isotropy of the CMB, just as many photons are scattered onto paths that lead to us. Thus, an increase in reionization has no affect on the overall CMB photon density or temperature. Reionization does affect the anisotropy spectrum, however. When photons are scattered off their paths towards us, they carry away their anisotropy information. The photons that are scattered in tend to replenish the anisotropy information at large scales (it is likely that they originated from the same “hot spot” or “cold spot” as the photons that were scattered out), while at small scales, the photons that are scattered in do not replenish the anisotropy (these photons did not originate from the same feature as the photons that were scattered out, so the overall effect is that “average” photons are scattered into the path). An increase in $\kappa_{rec}$ will enhance these reionization effects. This prediction is confirmed by figure 4.1, in which the model with an enhanced value of $\kappa_{rec}$ is unchanged at large scales, but is uniformly depressed at small scales.

Now that we have a greater understanding of the value of the power spectrum as a tool for determining the cosmological parameters, we return to our task of calculating the evolution of the CMB sky map in time.
Now that we have developed all the machinery that is necessary to calculate the CMB power spectrum from an input set of cosmological parameters, we can finally return to our main task of modeling how the CMB sky map evolves in time from the point of view of a stationary observer. This chapter is split into two sections — in the first, I discuss the extensions to the physical and mathematical framework of the previous chapter which will allow us to model the evolution of the CMB; while in the second, I explain my implementation of this extension.

5.1 Extension of CMB Theory

At first glance, it seems that we have everything we need, since we can already calculate the power spectrum at an arbitrary final time, $C_\ell(\tau_f)$. To create a representative theoretical sky map at a given time $\tau_f$, we can simply sample $a_{lm}$s from the distribution defined by $C_\ell(\tau_f)$, recalling that $C_\ell$ is the variance of the distribution from which the $a_{lm}$s are drawn at a given $\ell$:

$$a_{lm}(\tau_f) = r \sqrt{C_\ell(\tau_f)}$$  \hspace{1cm} (5.1)

Here, $r$ is a random number drawn from a Gaussian distribution with zero mean and unit variance (or simply a Gaussian random number). To generate a full sky map, the calculation in equation 5.14 must be iterated out to the maximum multipole we are interested in, and at each multipole, it must be iterated from $m = 0$ to $m = \ell$ (no independent information is contained in $a_{lm}$s with $m < 0$).

In our existing theoretical framework, therefore, we can calculate representative sky maps at any final time.\(^1\) However, the problem is that these sky maps are uncorrelated — a series of such maps at different times would not represent the evolution of the CMB from the point of view of a stationary observer. To truly model the time evolution of the CMB, then, we need a more complex method.

5.1.1 The $C_\ell$ Covariance Matrix

The solution to our problem of uncorrelated sky maps is the $C_\ell$ covariance matrix over the time steps we sample in our calculation. We define this covariance matrix as follows:

$$C^{ij}_\ell = \left< a_{lm}(\tau_f, \vec{x}) a^*_{lm}(\tau_f, \vec{x}) \right>$$  \hspace{1cm} (5.2)

\(^1\)This is only possible in the theoretical framework we have developed, it is not implemented in the existing version of CAMB, which always calculates the power spectrum at $\tau_0$. 

58
In this statement, I introduce the notation that $\tau_i$ is one of several times at which we would like to calculate the CMB sky map. From now on, we assume that the value of $\tau_i$ monotonically increases with $i$ and that $\tau_i = \tau_0$, the present time. If we have $N$ such final times, then $C^{ij}_\ell$ is an $N \times N$ matrix. This covariance matrix generalizes our notion of the power spectrum into an object that contains information about the evolution of the $a_{\ell m}$s from the point of view of a fixed observer.

We can very easily calculate this covariance matrix with only a small modification to the framework we developed in the previous chapter. First, we need to calculate $\Delta T_\ell(\tau_i, k)$ at every $\tau_i$ in which we are interested. This is done by setting the final time in the integration over the source function, equation 4.14:

$$
\Delta T_\ell(\tau_i, k) = \int_{\tau_i}^{\tau_f} S(\tau, k) j_i \left[ k(\tau_i^j - \tau) \right] d\tau
$$

The second step is to calculate the covariance matrix itself from the set $\Delta T_\ell(\tau_i, k)$. First, we should notice that our derivation of §4.2 would have proceeded in the same way if we had used equation 5.2 as our starting point for the derivation (equation 4.22). Carrying this through the derivation, we see that finding the $C_\ell$ covariance matrix only requires a slight change to equation 4.32:

$$
C^{ij}_\ell = (4\pi)^2 \int k^2 P_\ell(k) \left[ \Delta T_\ell(\tau_i^j, k) \Delta^* T_\ell(\tau_j^i, k) \right] dk
$$

Note that $C^{ij}_\ell = C^{ji}_\ell$ — the covariance matrix is symmetric. We are now required to calculate this entire covariance matrix, rather than just a single number, at each multipole $\ell$. The covariance matrix relates to the correlation matrix, which gives a dimensionless measure of the correlation between the distributions at two times $i$ and $j$:

$$
Cor^{ij}_\ell = \frac{C^{ij}_\ell}{\sqrt{C^{ii}_\ell C^{jj}_\ell}}
$$

Note that $Cor^{ij}_\ell \in [-1, 1]$. If the correlation very close to one, then the two distributions are highly correlated — $a_{\ell m}$s drawn from the distributions will be very similar (we expect this kind of behavior for two time steps that are close together — the CMB will not evolve appreciably over short time scales). If the correlation is zero, the two distributions are completely independent, and if the correlation is near negative one, samples drawn from the distributions will be anti-correlated.

At this point, we introduce matrix notation to simplify the calculations of the following sections. We will use boldface notation to denote the covariance matrix, $C_\ell$. In addition, we use the boldface notation $a_{\ell m}$ to denote a representative set $a_{\ell m}(\tau_i)$ across all sampled times — we should think of $a_{\ell m}$ as an $N$-dimensional column vector.\(^3\) Given this notation, we can rewrite equation 5.2 as follows:

$$
C_\ell = \langle a_{\ell m} a_{\ell m}^\dagger \rangle
$$

\(^2\)In typical runs, I use $N \approx 5$ or 10, the maximum size run was with $N = 101$.

\(^3\)From now on, we adopt the convention that boldface capital letters denote matrices and boldface lowercase letters denote column vectors. In addition, we use the notation $a_{\ell m}^i = a_{\ell m}(\tau_i^j)$. 


where the star denotes complex conjugation and transposition, so $a_{\ell m} a_{\ell m}^*$ is indeed an $N \times N$ matrix.

### 5.1.2 Measuring the Effect

In order to experimentally measure the time evolution effect, we would need to map the CMB at some time in the future (say 100 years from now), and then take the difference of this map with a map from the present era (from WMAP, for example). If we take the power spectrum of this “difference map”, we arrive at the power spectrum of the difference. We can perform this same operation in spherical harmonic space by taking the difference of corresponding $a_{\ell m}$s from the two times, and then taking the variance of the set of such differences at each $\ell$. Introducing the notation that $\Delta C_{ij}^{\ell} = \langle (a_{\ell m} - a_{\ell m}) (a_{\ell m} - a_{\ell m})^* \rangle$ (this is equivalent to the claim that $C_{\ell}$ is symmetric). We can therefore make a theoretical prediction of the power spectrum of the difference with our covariance matrix. This power spectrum of the difference is the interesting scientific quantity for our study. It is the signal that encapsulates the time evolution effect.

In order measure the effect, we need an experiment that is sensitive enough to see the power spectrum of the difference. To determine the parameters of such an experiment, we need to adapt the standard calculation for determining experimental error. The error on a power spectrum measurement (which we denote by $\sigma_C$) is given by (Bond et al., 1997):

$$
(\sigma_C)^2 \approx \frac{2}{(2\ell + 1) f_{\text{sky}}} \left( C_{\ell} + \frac{1}{w B_{\ell}^2} \right)^2
$$

where we are assuming an experiment with only a single frequency channel. In these equations, $f_{\text{sky}}$ is the fraction of the sky over which our experiment operates,$^4$ $\ell_{\text{s}}$ is the maximum multipole that the detector can resolve, $\theta_{\text{pix}}$ is the angular span of each pixel in the chosen sky pixelization scheme, and $\sigma_{\text{pix}}$ is the error on the measurement of each pixel, in $\mu K$.

To fully understand the relation between $\ell_{\text{s}}$ and $\theta_{\text{pix}}$, we need to introduce the beam profile, $P(\theta)$, which is a measure of the response of the detector to a point source as a function of the angle at which the detector views the source. For our purposes, the beam profile is given by a Gaussian, $P(\theta) = P_0 e^{-\theta^2 / 2\sigma_b^2}$, where we have introduced $\sigma_b$ as the standard deviation of the profile. By definition, $\ell_{\text{fwhm}} \equiv 1 / \sigma_b$. The resolution of a beam is defined by the angular scale of its full width at half-max, $\theta_{\text{fwhm}}$. This is the smallest angular scale that can be resolved by the detector. As its name

---

$^4$For a full-sky experiment, $f_{\text{sky}}$ is typically set to 0.75, to account for galactic interference.
suggests, $\theta_{\text{fwhm}}$ is the width of the Gaussian beam profile at the point at which its value is half of its maximum. Using this information, we can relate $\theta_{\text{fwhm}}$ and $\ell_s$:

\[
P(\theta_{\text{fwhm}}/2) = P_0/2
\]

\[
\exp\left(\frac{\theta_{\text{fwhm}}^2}{8\sigma_b^2}\right) = \frac{1}{2}
\]

\[
\sigma_b = \frac{1}{\ell_s} = \frac{\theta_{\text{fwhm}}}{\sqrt{8\ln 2}}
\]

for reference, $\theta_{\text{fwhm}}$ for WMAP is about 13 arcminutes, or 0.004 radians (Limon et al., 2006), corresponding to $\ell_s \approx 620$. Figure 5.1.2 shows a representative beam profile for a beam with $\ell_s = 2000$. We are completely free to choose $\theta_{\text{pix}}$, which specifies the pixelization scheme. However, it would be wise to choose $\theta_{\text{pix}} \approx \theta_{\text{fwhm}}$ to get the most out of our detector — we do not want to lose precision due to pixelization. We will assume the Nyquist sampling, which sets $\theta_{\text{pix}} = \theta_{\text{fwhm}}/2 = \sqrt{2\ln 2}/\ell_s$.

![Beam Profile](image)

**Figure 5.1:** A representative beam profile. Note how $\theta_{\text{fwhm}}$ gives a good representation of the angular resolution of the beam.

The measurement error, $\sigma_{\text{pix}}$, is calculated from the baseline detector sensitivity, which is measured in $\mu K \sqrt{s}$. If we denote this sensitivity by $\sigma_{\text{det}}$, then $\sigma_{\text{pix}}$ is:

\[
\sigma_{\text{pix}} = \frac{\sigma_{\text{det}}}{\sqrt{T_{\text{pix}} \sqrt{N_{\text{det}}}}}
\]

Where $t_{\text{pix}}$ (measured in seconds) is the time the detector can dedicate to each pixel, which is simply the total mission time ($t_m$) divided by the number of pixels in our pixelization scheme ($4\pi/(\theta_{\text{pix}})^2$), and $N_{\text{det}}$ is the number of independent detectors that make up the detector array.

Returning to equation 5.8, note that the $2/(2\ell + 1)$ prefactor is the inverse of the number of
independent $a_{\ell m}$s that we can measure at a given $\ell$. The equation quantitatively accounts for the cosmic variance (which we discussed in §1.6) through the $C_\ell$ term, while the $(wB_\ell^2)^{-1}$ term accounts for the experimental error due to the beam resolution.

We would like to use equation 5.8 to determine the parameters required of an experiment that can detect the power spectrum of the difference signal. We therefore make the following adaptation, denoting the error on an experimental measurement of $\Delta C_\ell^{ij}$ as $\sigma_\ell$:

$$\langle \sigma_\ell \rangle^2 \approx \frac{2}{(2\ell + 1) f_{\text{sky}}} \left( \frac{1}{wB_\ell^2} \right)^2$$  \hspace{1cm} (5.11)

where $w$ and $B_\ell^2$ are defined in the same way as they were in equation 5.8. We have dropped the cosmic variance term because we are taking the difference of $a_{\ell m}$s.

Equation 5.11 gives the error per $\ell$ on a measurement of the power spectrum of the difference. To tighten the errors, we bin them over a range of multipoles using the following equation for the final error:

$$\sigma_{\text{bin}} = \left[ \sum_{\ell_i}^{\ell_f} \frac{1}{\sigma_\ell^2} \right]^{-1/2}$$  \hspace{1cm} (5.12)

where $\ell_i < \ell_f$ are the boundaries of the bin.

To find parameters for an experiment that can see the power spectrum of the difference, we can simply alter the experimental parameters until the experiment yields errors that are on the same order as the signal. We will apply these techniques in the next chapter.

### 5.1.3 From $C_\ell$ to a Representative $a_{\ell m}$

It is an interesting theoretical exercise to create representative sky maps at each time step we consider, so that we can actually see how a representative sky evolves due to the time evolution effect. The covariance matrix $C_\ell$ allows us to do this, because it contains all the information necessary to produce a correlated representative set of $a_{\ell m}$s, as we will see below.

Our covariance matrix $C_\ell$ is symmetric and positive semi-definite. Hence, there exists an orthonormal matrix $M$ that diagonalizes $C_\ell$:

$$C_\ell = M D_\ell M^\dagger$$  \hspace{1cm} (5.13)

where $D_\ell$ is a diagonal matrix whose entries are the eigenvalues of $C_\ell$. Note that since $C_\ell$ is positive semi-definite, all its eigenvalues are non-negative.

I now claim that a representative sample $a_{\ell m}$ can be found via the following:

$$a_{\ell m} = M_{\ell} \left[ \sqrt{D_\ell} r \right]$$  \hspace{1cm} (5.14)

where $\sqrt{D_\ell}$ is the diagonal matrix with $\sqrt{D_\ell}^{ii} = \sqrt{D_\ell^{ii}}$ and $r$ is a column vector of Gaussian random
variables. To prove the claim, consider the following:

$$\langle a_\ell a^*_m \rangle = \left\langle \left( M_\ell \left[ \sqrt{D_\ell} r_1 \right] \right) \left( M_\ell \left[ \sqrt{D_\ell} r_2 \right] \right)^* \right\rangle$$

$$= \left\langle \left( M_\ell \left[ \sqrt{D_\ell} r_1 \right] \right) \left( \left[ r_2 \sqrt{D_\ell} \right] M_\ell^* \right) \right\rangle$$

$$= M_\ell \sqrt{D_\ell} \langle r_1 r_2^* \rangle \sqrt{D_\ell} M_\ell^*$$

$$= M_\ell D_\ell M_\ell^*$$

$$= C_\ell$$

In the second equality, I have noted that $\sqrt{D_\ell^*} = \sqrt{D_\ell}$, since $\sqrt{D_\ell}$ is a diagonal matrix with real entries. In the third equality, I note that the ensemble average only needs to be taken over the random vectors, since the $\sqrt{D_\ell}$ and $M_\ell$ matrices are constant. In the fourth equality, we use the fact that $\langle r_1 r_2^* \rangle = 1$, since all elements of $r_1$ and $r_2$ are drawn from the same distribution. The rest follows from the diagonalization of $C_\ell$, as established above.

We can therefore use equation 5.14 to calculate a correlated sample of $a_{\ell m}$s for all of our time steps.

## 5.2 Implementation

Implementing the procedures outlined in the above sections easily splits up into three separate tasks along the lines of the three previous sections. The first (and by far, most important and physically interesting) task is the calculation of the $C_\ell$ covariance matrix, which is performed by an altered version of CAMB, which is written in Fortran 90. The other two calculations (finding the parameters of the experiment needed to measure the effect and the calculation of representative maps) are analysis tasks that are separate from the main physical task of calculating the evolution of the CMB — hence they are performed in separate programs, which I have implemented in C++.

In the following sections, I document the implementation of these tasks in slightly greater detail. In order to understand the covariance matrix calculation, we need a baseline understanding of how the original CAMB source works, which we obtain in the next section. I make an effort to keep the discussion abstract, with as few direct references to the actual language implementations as possible. However, where appropriate, I describe the corresponding language-level data structures and routines in footnotes.

### 5.2.1 The Original CAMB Framework

At the highest level, CAMB is a program that takes a parameter file (typically called “params.ini”) as input and produces a theoretical power spectrum (in plot-able ASCII format) as output. This power spectrum is what we expect to see at the present time given the cosmological parameters specified in the parameter file. In addition to the cosmological parameters, this file allows the user

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5 Note that $\sqrt{D_\ell} r$ is a column vector.

6 CAMB also has the ability to produce FITS-formatted output files, but we will not use this facility in our extension.
to specify the maximum multipole to which the spectrum should be calculated, along with some other parameters that affect the accuracy of CAMB’s calculations.

We should note that CAMB makes heavy use of equation 2.70 to calculate times — for example, one of the first things CAMB does is calculate the age of the universe, $\tau_0$, by integrating this equation from $a = 0$ to $a = 1.$\footnote{CAMB stores the conformal age of the universe in a variable that is typically referred to as CP%tau0. Here, “CP” is an object of type CAMBParams (defined in modules.f90) which contains many fields concerning the parameters of the cosmological model. This particular variable carries a central role in our extension, as we will see in the next section.} We need to pay special attention to calculations in CAMB that involve time and those that make assumptions about the nature of $\tau_0$ (such as $a(\tau_0) = 1$), since our goal is to force CAMB to calculate the CMB spectrum at arbitrary final times. Also note that in the following explanation of the code, I only consider the calculation of scalar perturbations in a flat universe, since we do not account for non-flat models or non-scalar perturbations in our extension.

CAMB’s main computational routine\footnote{This routine is called “cmbmain”, defined in cmbmain.f90. It is called by CAMB_GetResults (camb.f90), which in turn is called by the driver program (inidriver.f90), which handles input of the parameter file and output of the final power spectrum.} performs four basic tasks: preliminary calculations, calculation of the source function, $S(\tau, k)$, calculation of the $\Delta_\ell(k)$s, and calculation of the power spectrum, $C_\ell$. We briefly discuss each in turn.

In its initialization phase, CAMB calculates several independent physical quantities that are needed for the source function calculation, and it initializes data structures that guide the rest of the calculations. CAMB does not calculate every $\Delta_\ell$, but rather samples $\ell$ values at which to perform the calculation, and after calculating the $C_\ell$ samples, it interpolates to all $\ell$ (this is legitimate because $C_\ell$ is a smooth function of $\ell$, without rapid oscillations). CAMB calculates the interpolation table (an array that lists the $\ell$ values at which the calculation is performed) in its initialization phase.\footnote{This calculation is preformed in “initlval” (modules.f90), and the interpolation array is typically referenced as “isamp%ld”, where isamp is an object of type “lSamples”.}

Next, the tables holding sampled values of the spherical Bessel functions (which are required to perform the line of sight integration in equation 4.14) are calculated.\footnote{Preformed by “InitSphereBessels” (bessels.f90).} CAMB begins doing physical calculations when it calculates the baryon sound speed, ionization fraction, and optical depth as functions of time.\footnote{In “inithermo”, defined in modules.f90.} All of these objects are calculated at a series of time samples from the big bang to $\tau_0$. The array of these time samples is also set up in this stage.\footnote{Performed by “SetTimeSteps” in modules.f90. The time step array is called “atau0” (contained in module “TimeSteps”), and it holds the conformal time of each time step CAMB considers in its calculations.}

In the second major phase, CAMB calculates the source function, $S(k, \tau)$.\footnote{This task is encapsulated by the “DoSourcek” function in cmbmain.f90.} In first-order perturbation theory, the separate $k$ modes are independent, so CAMB calculates every $k$ sample of $S(k, \tau)$ separately (in a loop). Prior to the actual calculation, CAMB sets up a table of $k$ values at which it will calculate the source function.\footnote{This is done in “DoSourcek”, in cmbmain.f90. Note CAMB’s convention of referring to the wavenumber as $q$, rather than $k$, because they are using the curved-space generalization of the wavenumber. We only consider flat universes in this study, so we can always consider $q$ to be the standard Fourier wavenumber.} Then, for each sampled $k$, CAMB evolves the source function from the initial conditions right after inflation to $\tau_0$, by evolving the set of coupled differential equations which we discussed in equations 3.44-3.47.\footnote{The function “GaugeInterface_ScalEV”, in cmbmain.f90, takes care of each evolution step.} We should recall that CAMB always assumes that $\tau_f = \tau_0$, and hence, its source function does not explicitly include the $e^{\tau_f}$ term — we
need to add it back in when we make our extension.

Equipped with the source function and the spherical Bessel functions, CAMB moves on to the calculation of the $\Delta T_\ell (k)$s, as given in equation 4.14. Again, the integration is performed for each $k$ sample separately. The routine that performs the integration for a given $k$ begins by pre-computing the quantity $j_\ell [k(\tau_f - \tau)]$ for every multipole $\ell$ being considered and for every $\tau$ sample used for the integration. We should also note that CAMB sets $\tau_f$ to $\tau_0$ in this calculation. After pre-computing the Bessel functions, the routine moves on to a loop over all multipoles, nested in which is a loop over time that actually performs the integration of equation 4.14.

The final step in the process is the calculation of the power spectrum. This is a fairly straightforward implementation of equation 4.32. The routine that implements the integration simply loops over the sampled $k$ values and performs the integration. At this point, CAMB interpolates the sampled $C_\ell$s so that it can output an entire spectrum, with $C_\ell$ listed for every $\ell$ up to the maximum.

This completes our brief tour of the data flow through the original CAMB framework. In the following section, we see that only a few slight changes are required to make the existing framework calculate the $C_\ell$ covariance matrix.

### 5.2.2 Alterations to CAMB

Before we begin describing the implementation of our extension, we should review our goals. Our program should take a user-specified list of times, given in years from the present (this helps with our intuition, since we can easily ask our program what the CMB will look like 100 years from now, for example). The output of the program should be a file that contains the contents of the $C_\ell$ covariance matrix for every multipole. To calculate the covariance matrix, we will need to adapt the CAMB source to use equation 5.4 instead of equation 4.32. Tracing the calculation backwards, we note that this requires us to calculate $\Delta T_\ell (k)$ not just at $\tau_0$, but also at all the specified future times. This requires us to use equation 4.14, which leaves the final time of the integration arbitrary, rather than using the original version, which sets $\tau_f$ to $\tau_0$. We also need to ensure that our program calculates the source function out to the latest time in which we are interested (call this time $\tau_{max}$), rather than simply to $\tau_0$. The implementation should be as modular as possible, and any alterations to the original source should be conditionally compiled.

Rather than adjust the existing data structures that hold the $\Delta T_\ell$ and $C_\ell$ information, we create

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16 Internally, CAMB stores the $\Delta T$s in an array called “ThisCT%Delta_p_1_k”. ThisCT is of type “ClTransferData”. The Delta_p_1_k array has three indices — the first is the source type (we are only interested in scalar sources, so we only care about the case where this index is one), the second is the multipole index (feeding this index to the lSamp%l array gives the actual multipole represented), and the third is the wavenumber index (the array “ThisCT%q_int” is the map from this index to the actual $k$-value — note that this sampling is different from the sampling used for the source integration; the source function is interpolated as necessary to determine values for the new sampling).

17 Ultimately, for our purposes, this is done by the “DoFlatIntegration” method in cmbmain.f90 (again, we are only considering the flat-universe case), which is called by “DoSourceIntegration”.

18 The “CalcScalCls” function in cmbmain.f90. The calculated $C_\ell$s are stored in the “iCl_scalar” array, where the “i” is meant to remind us that these are the pre-interpolation samples of the power spectrum.

19 CAMB places the full (interpolated) $C_\ell$ spectrum in an array called “Cl_scalar”.

20 All alterations made to the original CAMB files are conditionally compiled using the C preprocessor. The accompanying makefile will compile CAMB in its original form unless the “FUTURE_TIMES” macro is set, in which case our extended version of CAMB (called “camb_future”) is compiled.
CHAPTER 5. MODELING CMB EVOLUTION

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a completely new module\textsuperscript{21} that contains arrays for $\Delta T_\ell (k, \tau)$ and $C_\ell$ that are appropriate for our more expansive needs.\textsuperscript{22} This new module also reads input from the params.ini file, in which the user is allowed to specify the requested time steps.\textsuperscript{23} It then converts the “years from the present” information provided by the user into the corresponding conformal times and scale factors. We maintain the convention that the value of the scale factor at the present time is unity, so to perform this conversion for a given number of years from the present $\Delta t$, we use a modified form of equation 2.71:

$$\Delta t = \int_1^{a_f} \frac{1}{a' H_0 \sqrt{\Omega_r,0 + a^2 \Omega_m,0 + a^4 \Omega_k,0 + a^4 \Omega_\Lambda}} da'$$

(5.16)

where we solve for the final scale factor $a_f$ by performing the numerical integration until the sum is equal to the requested $\Delta t$. With this $a_f$, we then calculate the corresponding $\tau_f$ using equation 2.70. Our module now knows the conformal time and scale factor for each time step.\textsuperscript{24}

The most insidious element of this implementation is the method for convincing CAMB to calculate the source function out to $\tau_{\text{max}}$. To do this, we “trick” CAMB into thinking that $\tau_0$ is the latest future time we wish to consider, by setting its internal $\tau_0$ variable\textsuperscript{25} to $\tau_{\text{max}}$.\textsuperscript{26} The existing CAMB code already allocates time steps out to $\tau_0$ and calculates the source function at all such time steps, so altering the internal $\tau_0$ in this way achieves our goal with no extra effort on our part — CAMB will happily calculate $S(\tau, k)$ out to $\tau_{\text{max}}$. We need to be careful with this approach, however. Note that we will still use $\tau_0$ to denote the present conformal age of the universe — changing CAMB’s internal value is really just a trick to which we should not assign physical significance. Also, we need to watch out for places in the code where special meaning is given to $\tau_0$ — for instance, the scale factor at CAMB’s internal $\tau_0$ is no longer unity.

Now that we have the source function out to $\tau_{\text{max}}$, the rest of the implementation is quite straightforward. The next step is to calculate $\Delta T_\ell (k, \tau_f)$ for every $\tau_f$ under consideration. To do this, we use equation 4.14:

$$\Delta T_\ell (\tau_f, k) = \int_{\tau_0}^{\tau_f} S(\tau, k) j_\ell [k(\tau_f - \tau)] d\tau$$

(5.17)

This requires us to modify the existing CAMB routine\textsuperscript{27} by adding another loop inside the loop over multipoles (but outside the integration loop). This loop is over all the final times ($\tau_f$) under consideration. Note that we have to change the pre-computation of the Bessel functions because the argument is now $k(\tau_f - \tau)$ instead of $k(\tau_0 - \tau)$. In addition, we need to change the upper limit

\textsuperscript{21}This module is called “future”, and is implemented in future.f90.

\textsuperscript{22}The data structures are contained in a new type, “FutureTimes”. We create an instance of this type, called “FT”, as a member of module “ModelParams”, which is defined in modules.F90.

\textsuperscript{23}The user must specify “num_times” in the parameter file (it is recommended that the number of times be kept around ten for the sake of computation time) as well as “delta_time(i)”, for $i \in [1, \text{num_times}]$, given in years from the present. Internally, we store the number of future time steps in “FT%NumTimes” and the time steps in “FT%DeltaTimes”.

\textsuperscript{24}In the implementation, the “FT%Times” and “FT%ScaleFactors” arrays store this information. Their indices range from zero (the present time), to the user specified num_times. All time-ranging indices in module future share this same convention.

\textsuperscript{25}Recall that this variable is CP%tau0.

\textsuperscript{26}In the implementation, $\tau_{\text{max}}$ is given by FT%Times(FT%NumTimes). The actual assignment takes place in the CAMB-Params_Set function in modules.F90.

\textsuperscript{27}DoFlatIntegration, in cmbmain.F90.
of the integration from $\tau_0$ to $\tau_f$. The result of this calculation is an array containing the $\Delta \ell(k, \tau_f)$ for all our $\tau_f$s. At this point, we need to recall that the source function we derived (equation 4.17) includes a prefactor of $e^{\kappa_f}$ which the standard CAMB framework does not include, since it assumes $\tau_f = \tau_0$. Notice that this prefactor can be factored completely out of the $\Delta \ell$ integration (4.14), leaving the prefactor multiplying the quantity that we just calculated. To get the proper $\Delta \ell(k, \tau_f)$, therefore, we multiply by our result (as outlined above) by $e^{\kappa_f(\tau_f)}$.

Finally, we are in a position to calculate the covariance matrix, $C_\ell$, using equation 5.4. We simply need to add a double loop over time around the existing integration loop. Again, we use our own data structure to store the covariance matrix. Recall that CAMB does not calculate every multipole, but rather interpolates the full power spectrum after the fact from a sampling of the power spectrum — what we have calculated thus far is a sample of $C_\ell$, we did not calculate it at every multipole. All elements of the covariance matrix should be a smooth function of multipole (this is confirmed by reviewing the results of the calculation), so we can interpolate each element of $C_\ell$ individually to obtain our final result, the covariance matrix at every multipole out to the user-specified maximum. Our add-on module includes a routine that writes the covariance matrix out to an ASCII file, as well as the correlation matrix, $\text{Cor}_\ell$, and the power spectrum of the difference matrix, $\Delta C_\ell$.

We must be careful when we give units to the $C_\ell$ matrix. The elements of the covariance matrix are inherently dimensionless, so to give them units after the calculation, we must scale them. In the standard power spectrum calculation, CAMB simply multiplies the dimensionless $C_\ell$ by the squared average CMB temperature at the present time (typically in $\mu K^2$). However, our $C^i_\ell$ are formed by integrating over the product of $\Delta \ell$s at two times, $\tau_i$ and $\tau_j$, at which the average temperature of the CMB may be quite different. Hence, to give $C^i_\ell$ units, we must multiply by the following quantity:

$$\frac{T^2_0}{a(\tau_i)a(\tau_j)}$$

which accounts for the time evolution of the average CMB temperature. We can then calculate the power spectrum of the difference, using equation 5.7, with these “scaled” elements of the covariance matrix.

As a final note, we must account for the optimizations present in CAMB that may interfere with our calculation. In many of CAMB’s routines that involve loops over time, the loops are cut off early because the behavior at late times is assumed to be simple (the source function is simply set to zero at late times, for example). However, in our extension, we are very interested in this late-time

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28 In the implementation, this is a three dimensional array — “FT%Delta_1_k_tau”. The first index represents the multipole ([Samp%l translates this to the actual multipole), the second index represents the wavenumber index (ThisCT%q_int translates to the actual wavenumber), and the third index represents the final time (FT%Times at this index gives the represented conformal time).

29 We perform the calculation in the subroutine “CalcScalCls” in cmbmain.F90.

30 A three dimensional array, “FT%Cl_covariance”, is used. The first index is the multipole index, and the second and third are time indices (again, representing the elements of FT%Times).

31 We use “FT%Cl_covariance” to store the covariance matrix for every $\ell$ — the first index into this array is the actual value of the multipole.

32 Note that since these matrices are all symmetric, we only need to output the elements with $j \geq i$.

33 Such as CalcScalarSources and DoFlatIntegration.
behavior, so we remove all such time-cutoffs, forcing CAMB to actually perform the full physical
calculations even at the late times we consider.

5.2.3 ADDITIONAL PROGRAMS

Two additional programs perform calculations on the covariance matrix information after it is pro-
duced by our new version of CAMB. Both programs are fairly straightforward and require little
documentation here.

The first, called “future_alms”, calculates the representative \( a_{\ell m} \)s as described in section 5.1.3.
To perform the linear algebra operations, it uses the Template Numerical Toolkit and JAMA/C++
libraries.\(^{34}\) To generate uniform random numbers, future_alms uses the R250 pseudo-random num-
ber generator;\(^{35}\) it converts these to Gaussian random numbers with the Box-Muller transform. The
program creates a binary FITS file for every time step, and writes the \( a_{\ell m} \)s for each time step to the
appropriate file. These files are in the proper format for reading by SYNFAST, which can take them
and create the corresponding sky map in HEALPix\(^{36}\) format (this inverts the spherical harmonic
transform). This procedure allows us to actually visualize a representative series of correlated sky
maps that show how the CMB photosphere evolves in time. In addition to creating these FITS files,
future_alms performs sanity checks on our power spectra and power spectra of the difference by
re-calculating these quantities based on the sample \( a_{\ell m} \)s the program calculates.

The second program, called “get_error”, performs the error calculation outlined in section 5.1.2.
The user has the ability to set the parameters of the experiment as command-line arguments. The
program typically operates on a single element of the \( \Delta C_{\ell} \) matrix, and since we are interested in
the power spectrum of the difference because it tells us the measurability of our effect, we will ex-
clusively use \( \Delta C^{0}_{\ell} \) elements, which give the power spectrum of the difference between time \( \tau_j \)
and the present. In addition to calculating the error per \( \ell \), \( \sigma_{\ell} \), get_error bins the error over logarithmic
intervals (the scale of the interval can be set by the user). The program writes this information in
ASCII files, and their content can then be plotted.

\(^{34}\)http://math.nist.gov/tnt/
\(^{35}\)http://www.taygeta.com/random.xml
\(^{36}\)http://healpix.jpl.nasa.gov/healpixSoftwareGetHealpix.shtml
CHAPTER 6
RESULTS

Now that all the theoretical and computational groundwork is in place, we now have the ability to explore the time evolution of the CMB. In this chapter we will examine the large-scale effects that occur at very large time steps into the future, the small scale-effects that occur in the near future, and the evolution in-between. We conclude by determining the parameters required of an experiment designed to measure the time evolution effect.

The main results run was performed using the best-fit WMAP ΛCDM cosmology (which assumes a flat universe) which we introduced in §4.3. Throughout this discussion, a “time step” is defined by a given amount of physical time into the future from τ₀ (one hundred years, one million years, ten billion years, etc.). Table 6.1 gives the corresponding physical time, conformal time, and scale factor for each time step (as calculated by our model). This table is a useful reference for the discussion in this chapter.

<table>
<thead>
<tr>
<th>Time Step (Years)</th>
<th>Time (Years)</th>
<th>Conformal Time (Mpc)</th>
<th>Scale Factor</th>
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<td>Present</td>
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<td>1.0000000000</td>
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</tbody>
</table>

Table 6.1: The time steps used in the main results run. The physical time, conformal time, and scale factor for each time step are given.

Table 6.1 has some interesting properties in its own right, which we should note before diving into the CMB results. First, it is important to notice how little the conformal time and scale factor change over short time scales. It takes the scale factor, for example, more than one hundred million
years to increase by one percent from its present-day value. A recurring theme of this chapter is
that relatively small changes in time (compared to the age of the universe) produce correspondingly
small effects. It is also interesting to note the relationship between the physical time and conformal
time in table 6.1. Over the time steps shown in the table, the physical time more than quadruples
while the conformal time increases only by about thirty percent. This is due to the fact that the
growth of conformal time is tempered by the scale factor.

6.1 Power Spectrum Analysis of Late-Time Effects

We begin our discussion of the CMB results by observing changes in the power spectrum at ex-
tremely large time steps into the future. Figure 6.1 shows the temperature angular power spectrum
at the present time and at several large time steps.

The most obvious effect is the depression of the overall anisotropy power as time progresses,
due to the reduction of the average CMB temperature with time. The CMB temperature falls like
$1/a$, as we discussed in §2.1.1, which implies that the anisotropy power drops as $1/a^2$.

The next effect is the fact that the features of the power spectrum (the peaks and troughs) uni-
formly shift to smaller angular scales (larger multipoles) as time progresses. We expected this as
well, based on our argument in §1.7 — the recession of the CMB photosphere means that photons
arriving from features of fixed conformal size (wavenumber) will arrive at increasingly smaller
angular scales as time progresses. This effect is more easily seen in figure 6.2, where the power
spectrum $10^{10}$ years from now is plotted at the same relative scale as the power spectrum of today.
In addition to the shift to smaller scales, figure 6.2 makes it clear that the low-$\ell$ tail of the power
spectrum is enhanced (as a fraction of the peak power) at the later time. This is exactly what we
expect in a $\Lambda$ dominated universe, based on our discussion of the integrated Sachs-Wolfe effect in
§4.2. At the present time in our best-fit model, we are just entering the era of $\Lambda$ dominance, so we
expect that as time progresses and the transition to $\Lambda$ dominance continues, a late-time ISW effect
will enhance the low-$\ell$ tail of the CMB power spectrum.

We can confirm this by considering a flat universe with no cosmological constant — i.e., a flat
universe dominated by cold dark matter. Figure 6.3 is the analogue of figure 6.2 for a CDM uni-
verse, showing the power spectrum at $\tau_0$ and the power spectrum ten billion years hence on the
same relative scale. The figure shows that there is no change in the relative scale of the low-$\ell$ tail in
the CDM model, confirming our suspicion that the low-$\ell$ enhancement in our base model is due to
the presence of dark energy and the ISW effect it causes at late times.

Returning to our standard WMAP best-fit $\Lambda$CDM universe, we note that in figure 6.1, the low-$\ell$
tail does not increase on an absolute scale as time progresses, but rather only increases as a fraction
of the total peak, as can be seen in figure 6.2. In fact, the data from this run show that the low-$\ell$
region of the power spectrum is not larger than the present-day level for any of the time steps
considered. This observation leads us to note that the overall depression of the power spectrum
due to the expansion of the universe outpaces the enhancements at low $\ell$ due to the transition to $\Lambda$
dominance. Additionally, the simulations suggest that at some point in the future, the low-$\ell$ peak

---

1I used a model with $\Omega_b = 0.17$, $\Omega_c = 0.83$, $\Omega_\Lambda = 0$, and $H_0 = 72 \text{ km/s/Mpc}$.
2This fact was confirmed simply by checking for $C_\ell^{ij} - C_\ell^{00} < 0$ for $\ell < 200$, at all time steps $i$. 
will in fact become larger than the first acoustic peak. Figure 6.4 confirms this suspicion. The figure shows the power spectra for several extremely late time steps, all of which have been relatively normalized to their first acoustic peak to eliminate the overall temperature depression. The figure indicates that the low-$\ell$ tail overtakes the first acoustic peak sometime between twenty and thirty billion years from now.

Next, we take a more quantitative look at the effects on the power spectrum. Table 6.2 gives the position and amplitude of the first four acoustic peaks of the power spectrum. This information again confirms both the depression of power due to the overall in the average CMB temperature and the shift to smaller angular scales. Recall that in §1.7, we predicted that a shift to smaller scales would indeed occur as time progresses, due to the recession of the surface of last scattering. This shift to smaller scales should be linear in $\tau$, since perturbations of wavenumber $k$ manifest themselves as anisotropies at multipole $\ell = 2\pi k x_{rec}$, where $x_{rec} = \tau_0 - \tau_{rec}$ is the comoving distance to the surface of last scattering ($\tau_{rec}$ is the conformal time of last scattering). Hence, a feature at a given multipole $\ell$ shifts from its present day multipole $\ell_0$ according to the following relationship:

$$\frac{\ell_0}{\ell_i} = \frac{\tau_0 - \tau_{rec}}{\tau_i - \tau_{rec}}$$

(6.1)

We can apply this formula to our data, and check to see how close this naïve prediction gets us to the actual shift of the power spectrum produced by the model. The results of such a check are shown in table 6.3. For each peak, I have listed the calculated $\ell$-position based on the run data (as in table 6.2) and then an expected $\ell$-position, calculated from the position of the corresponding peak at $\tau_0$ using equation 6.1. The table shows that the simple argument of §1.7 holds up quite well, as its predictions are very close to the values predicted by the full machinery of our evolution model (in fact, the two values are almost always identical).

<table>
<thead>
<tr>
<th>Time Step</th>
<th>1$^{st}$ peak</th>
<th>2$^{nd}$ peak</th>
<th>3$^{rd}$ peak</th>
<th>4$^{th}$ peak</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\ell$ power</td>
<td>$\ell$ power</td>
<td>$\ell$ power</td>
<td>$\ell$ power</td>
</tr>
<tr>
<td>Present</td>
<td>220 5626</td>
<td>536 2526</td>
<td>815 2450</td>
<td>1129 1209</td>
</tr>
<tr>
<td>$1.0 \times 10^7$ Years</td>
<td>220 5617</td>
<td>536 2522</td>
<td>815 2446</td>
<td>1129 1208</td>
</tr>
<tr>
<td>$1.0 \times 10^8$ Years</td>
<td>220 5543</td>
<td>537 2489</td>
<td>817 2414</td>
<td>1131 1192</td>
</tr>
<tr>
<td>$5.0 \times 10^8$ Years</td>
<td>222 5229</td>
<td>541 2348</td>
<td>823 2277</td>
<td>1141 1124</td>
</tr>
<tr>
<td>$1.0 \times 10^9$ Years</td>
<td>224 4865</td>
<td>547 2184</td>
<td>832 2117</td>
<td>1152 1046</td>
</tr>
<tr>
<td>$2.0 \times 10^9$ Years</td>
<td>229 4221</td>
<td>557 1895</td>
<td>848 1838</td>
<td>1175 907</td>
</tr>
<tr>
<td>$4.0 \times 10^9$ Years</td>
<td>236 3202</td>
<td>576 1437</td>
<td>876 1395</td>
<td>1214 688</td>
</tr>
<tr>
<td>$6.0 \times 10^9$ Years</td>
<td>243 2447</td>
<td>593 1098</td>
<td>901 1066</td>
<td>1249 526</td>
</tr>
<tr>
<td>$8.0 \times 10^9$ Years</td>
<td>249 1879</td>
<td>607 844</td>
<td>923 818</td>
<td>1279 404</td>
</tr>
<tr>
<td>$1.0 \times 10^{10}$ Years</td>
<td>254 1448</td>
<td>620 650</td>
<td>943 630</td>
<td>1306 311</td>
</tr>
</tbody>
</table>

Table 6.2: The positions and amplitudes of the first four peaks of the power spectrum as a function of time into the future. Power is given as $(\ell(\ell+1)C_\ell)/2\pi$. Note that there is no noticeable difference from the present for all time steps before ten million years, hence they are not listed in the table. This table shows both the overall depression of the anisotropy power and the shift to smaller angular scales.

$^3$Figure 1.7 illustrates this effect.
Figure 6.1: The temperature angular power spectrum of the CMB at several representative time steps into the future. Note the three major effects: (1) The power spectrum amplitude drops off due to the $1/t$ scaling of the CMB temperature; (2) The features shift to smaller angular scales due to the recession of the surface of last scattering; (3) The low-$\ell$ tail becomes enhanced compared to the peak due to the integrated Sachs-Wolfe effect caused by the recent shift to dark-energy dominance.

Figure 6.2: A normalized power spectrum plot accentuating the shift to smaller angular scales and the ISW effect at extremely late times. The left axis corresponds to the present-day plot, and the right-axis corresponds to the late-time plot.
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Figure 6.3: This plot shows the power spectrum at $\tau_0$ and $10^{10}$ years into the future in a CDM universe at the same relative scale. Note that in the CDM universe, there is no ISW enhancement of the low-$\ell$ region as there is in a $\Lambda$CDM universe.

Figure 6.4: Anisotropy power for several extremely late time steps, normalized to the power of the first acoustic peak. The low-$\ell$ tail, enhanced by the late time integrated Sachs-Wolfe effect, eventually overtakes the first acoustic peak in power.
Table 6.3: Predicting the peak shift. At each time step $i$, the conformal time, $\tau_i$, is given. For each peak of the power spectrum, the “A” column gives the “actual” peak as calculated from the model; while the “P” column gives the peak that is “predicted” by the simple formula $\ell_i = \ell_0 \left( \frac{\tau_i - \tau_{rec}}{\tau_0 - \tau_{rec}} \right)$. The prediction and the model results are in close alignment.

6.2 Evolution of the Sky Map

Next, we turn to the goal that motivated this entire project — visualization of the evolution of the CMB photosphere for a stationary observer. Using the theoretical method we developed in section §5.1.3, we sampled correlated $a_{\ell m}$s across all the time steps we considered for this run. The $a_{\ell m}$s for a given time step were then transformed into a CMB sky map.\(^4\) The end result of such a run, therefore, is a representative sequence of correlated CMB sky maps that portray the evolution of the CMB for a fixed observer. We should note that there is randomness involved — our sequence of maps does not predict exactly how the CMB will evolve, but rather, it is a sample of the possible evolutionary trajectories allowed by the $C_\ell$ covariance matrix. This is the generalization of standard CMB modeling techniques, in which a representative sky map can be produced from a power spectrum — such a sky map is simply one of many possible sky maps described by the power spectrum.

Turning now to the actual sky maps produced by our model run, we should first note that changes in the sky map are not visible until we go out approximately $10^8$ years into the future. At smaller time steps, the anisotropy maps are virtually indistinguishable from the anisotropy map at $\tau_0$. Figures 6.5-6.10 show a possible CMB sky map evolution sequence, from the present time to forty billion years into the future. The increased power at large angular scales due to the late-time ISW effect is clearly visible at the later time steps. The shift of the first acoustic peak to smaller angular scales is also visible, although the effect is more easily seen in figures 6.2 and 6.4.

\(^4\)Sky maps are created using “SYNFAST” and “MAP2GIF”, members of the HEALPix package.
Figure 6.5: Representative temperature anisotropy map of the CMB at the present time in a $\Lambda$CDM universe. Temperature is given in Kelvin.

Figure 6.6: Representative temperature anisotropy map of the CMB $10^8$ years from now in a $\Lambda$CDM universe. Temperature is given in Kelvin.
Figure 6.7: Representative temperature anisotropy map of the CMB $10^9$ years from now in a $\Lambda$CDM universe. Temperature is given in Kelvin.

Figure 6.8: Representative temperature anisotropy map of the CMB $5 \times 10^9$ years from now in a $\Lambda$CDM universe. Temperature is given in Kelvin.
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Figure 6.9: Representative temperature anisotropy map of the CMB $10^{10}$ years from now in a $\Lambda$CDM universe. Temperature is given in Kelvin.

Figure 6.10: Representative temperature anisotropy map of the CMB $4 \times 10^{10}$ years from now in a $\Lambda$CDM universe. Temperature is given in Kelvin.
6.3 Difference Analysis of Near-Time Effects

For small time steps, the evolution effects are not strong enough to manifest themselves as visible differences in the power spectra. Hence, to study near-time CMB evolution effects, we must look at the difference of the sky map with the present-time sky map, and the power spectrum of this difference. Throughout this discussion, whenever we discuss the power spectrum of the difference “at” a given time step, it is implied that the difference is taken with the present-day map.

Figure 6.11 shows the difference between the CMB sky map one hundred years into the future and the CMB sky map today. Note that these two maps are part of the correlated sequence of maps we discussed in the previous section. To produce a difference map, we take the corresponding sampled \( a_{\ell m} \)s at the two time steps of interest and subtract them. This produces a set of \( a_{\ell m} \)s that can be transformed into a map, just as we did before, yielding the expected CMB difference. Of interest is the fact that this map has no large-scale structure and consists only of very fine-grained “noise”. We should note that the anisotropies in this temperature difference signal are on the order of \( 10^{-10} \) Kelvin — the rms anisotropy is \( 8 \times 10^{-11} \). This value is six orders of magnitude smaller than the CMB anisotropies we see today. Therefore, in order to detect the time evolution effect, we would need to be sensitive to signals that are significantly smaller than the anisotropy signals we are used to.

The power spectrum of the difference for the 100 year time step, and for several later time steps, is given in figure 6.12 — note that the vertical axis is logarithmic (unlike our previous power spectrum plots). At early times, the difference spectrum has a single peak centered at \( \ell \approx 750 \), which is between the second trough and third peak of the power spectrum. At higher multipoles, the effect of the damping tail is so great that the power spectrum of the difference is washed out (there is very little anisotropy to begin with at such multipoles, and hence, there can be no difference signal). The strength of the power spectrum of the difference increases linearly with time, and slowly rises with multipole until the damping tail begins to dominate. The most important item to take from these difference spectra is how small the signal is in comparison to the regular power spectrum signal.\(^5\) This fits well with our intuition — we do not expect the CMB map to change appreciably over such short time scales.

Late time steps demonstrate interesting behavior in the power spectrum of the difference with the present, as we can begin to see from the \( 10^9 \) and \( 10^{10} \) year time steps shown in figure 6.12. Figure 6.13 shows the power spectrum of the difference for several time steps beginning with \( 10^8 \) years, at which time the power spectrum of the difference still exhibits its “early-time” form, with a single peak at \( \ell \approx 750 \). At later time steps, we see that the power spectrum of the difference takes on the appearance of a typical power spectrum, in both shape and in scale. Recall that the power spectrum of the difference is given by \( C_\ell^{00} + C_\ell^{jj} - 2C_\ell^0 \). We will see in the next section that the covariance with the present \( (C_\ell^0) \) gets very small as time progresses (due to the fact that the two times become less correlated), and hence the power spectrum of the difference reduces to \( C_\ell^{00} + C_\ell^{jj} \). The power spectrum of the difference therefore increases to a peak level at around \( 10^{10} \) years and then tapers.

\(^5\)It is important to remember that it is the temperature (in \( \mu K \)), and not the power (in \( \mu K^2 \)), that is measured by an experiment. While the power spectrum of the difference at 100 years is about 12 orders of magnitude smaller than the power spectrum today, the scientifically important scale is the six orders of magnitude separating the temperature anisotropies, as we discussed earlier.
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Figure 6.11: The difference between the representative anisotropy maps at the 100 year time step and the present.

Figure 6.12: The power spectrum of the difference for several time steps on a logarithmic scale, showing the evolution of the difference spectrum with time. The difference spectrum maintains the same basic shape, but becomes more pronounced as time progresses.
off as the power spectrum at the late time, \( C_{\ell} \) drops away due to the overall temperature scaling. The power spectrum of the difference then asymptotically approaches the power spectrum at the present, \( C_{\ell}^0 \).

![The Power Spectrum of the Difference at Late Times](image)

**Figure 6.13:** The power spectrum of the difference for late time steps. The difference signal increases until the \( 10^{10} \) year step, after which it decreases back down to the level of the power spectrum at the present.

### 6.4 Sanity Checks

Now that we are familiar with some of the results of our study, we should pause to make sure that our methods are reasonable and to double-check some of our conclusions.

#### 6.4.1 The Covariance Matrix

First, we review our claim that arbitrary elements of the covariance matrix are smooth functions of \( \ell \), which justifies the fact that we only calculate an \( \ell \)-sampling of the covariance matrix, and interpolate each element to all other multipoles. Figure 6.14 shows the covariance with \( \tau_0 \) of several future time steps, plotted in the same figure. The figure confirms that the covariances are indeed smooth functions of \( \ell \). Note that the covariance at near-future times is very similar (nearly identical, actually) to the power spectrum at \( \tau_0 \). This is yet more confirmation that the time evolution effect is very small for near-future times. At later times, the covariance with the present becomes smaller, indicating that the distribution of \( a_{\ell m} \) becomes less and less correlated with the distribution at the present as time progresses, which agrees with our intuition.

To continue our study of the covariance matrix, we look at a specific \( \ell \)-value of the \( C_{\ell} \) matrix,
and use it as a case study to explore the linear algebra we developed in section 5.1.3. For this study, we will consider the $\ell = 200$ element (near the first acoustic peak), for a run with only four time steps. The indices of the $C_\ell$ matrix correspond in ascending order to the time steps considered — the present (index 0), 100 years (index 1), $10^8$ years (index 2), and $10^{10}$ years (index 3). As calculated by our model, the $C_{200}$ matrix for this run is:

\[
C_{200} = \begin{pmatrix}
8.609 \times 10^{-13} & 8.609 \times 10^{-13} & 7.786 \times 10^{-13} & 5.966 \times 10^{-16} \\
8.609 \times 10^{-13} & 8.609 \times 10^{-13} & 7.786 \times 10^{-13} & 5.966 \times 10^{-16} \\
7.786 \times 10^{-13} & 7.786 \times 10^{-13} & 8.475 \times 10^{-13} & 5.859 \times 10^{-16} \\
5.966 \times 10^{-16} & 5.966 \times 10^{-16} & 5.859 \times 10^{-16} & 2.007 \times 10^{-13}
\end{pmatrix}
\] (6.2)

This symmetric covariance matrix contains all the features we expected to see. The fact that the upper $2 \times 2$ block shares the same value indicates that the present time step is completely correlated with the 100 year time step.\footnote{Note that not all the calculated digits are shown — there are slight differences at higher precision, which account for the effects at the 100 year time step we discussed in the last chapter.} Note how the diagonal elements (the power spectrum at each time step) decrease with time, and how the covariance with zero (the $C_{200}^{0j}$ elements) drops off dramatically at the highest time step — indicating that the time steps become highly uncorrelated. These
The next step in the process is to diagonalize the covariance matrix \( \mathbf{C}_\ell = \mathbf{M}_\ell \mathbf{D}_\ell \mathbf{M}_\ell^* \). To save space, I will not show the \( \mathbf{M}_{200} \) and \( \mathbf{D}_{200} \) matrices here. Continuing our tour of §5.1.3, we note that a column vector of correlated \( a_{\ell m} \)s is sampled via equation 5.14: 

\[
\mathbf{a}_{\ell m} = \mathbf{M}_\ell \left( \sqrt{\mathbf{D}_\ell} \right) \mathbf{r}.
\]

(6.4)

where \( \sqrt{\mathbf{d}_\ell} \) is a column vector whose entries are the elements of \( \sqrt{\mathbf{D}_\ell} \). Hence, we can think of \( \mathbf{M}_\ell \sqrt{\mathbf{d}_\ell} \) as the generator matrix that transforms a vector of Gaussian random variables into a correlated set of \( a_{\ell m} \)s. For our study, the generator matrix is:

\[
\mathbf{M}_{200} \sqrt{\mathbf{d}_{200}} = \begin{pmatrix}
-1.287 \times 10^{-13} & -1.255 \times 10^{-7} & -1.377 \times 10^{-10} & 9.193 \times 10^{-7} \\
1.287 \times 10^{-13} & -1.255 \times 10^{-7} & -1.377 \times 10^{-10} & 9.193 \times 10^{-7} \\
-1.142 \times 10^{-19} & 2.615 \times 10^{-7} & -7.424 \times 10^{-11} & 8.827 \times 10^{-7} \\
2.922 \times 10^{-23} & -3.386 \times 10^{-11} & 4.480 \times 10^{-7} & 7.115 \times 10^{-10}
\end{pmatrix}
\]

(6.5)

It is important to note that the first two rows of this generator matrix are nearly identical, which causes the sampled \( a_{\ell m} \)s for the present and the 100 year time step to be nearly identical as well. This exploration makes us more confident that our covariance matrix method of \( a_{\ell m} \) sampling is working correctly.

### 6.4.2 Recalculated Spectra

Our next task is to ensure that our sampled \( a_{\ell m} \)s actually follow the theoretical power spectra at their respective time steps. Our program that samples the \( a_{\ell m} \)s also recomputes the power spectrum at each time step. Figure 6.15 shows the recomputed power spectrum as data points with the original theoretical power spectrum overlaid for comparison for the \( 10^8 \) year time step. This figure shows that the sampled \( a_{\ell m} \)s do indeed follow the power spectrum at their time step. This observation is not as trivial as it seems, since the \( a_{\ell m} \)s are not directly calculated from the power spectrum, but by the generator matrix method we explored in the previous section and introduced in §5.1.3. Hence, the fact that figure 6.15 shows that the \( a_{\ell m} \)s actually follow the power spectrum should give us confidence in the theory we developed in §5.1.3.

In the same vein, we can ensure that our theoretical calculation of the power spectrum of the difference is legitimate. To do so, we recalculate it directly from the \( a_{\ell m} \)s, by applying the first step
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Figure 6.15: The theoretical power spectrum at $10^8$ years compared to the power spectrum that is recalculated by taking the variance of the sampled $a_{lm}s$. The agreement is encouraging.

Figure 6.16: The theoretical power spectrum of the difference at 100 years compared to the power spectrum of the difference that is recalculated from the sampled $a_{lm}s$ at 100 years and the present. Again, the agreement is encouraging.
of equation 5.7:
\[
\Delta C_{ij}^{\ell} = \left( a_{ijm}^\dagger - a_{im}^\dagger \right) \left( a_{ijm}^\dagger - a_{im}^\dagger \right)^* \tag*{(6.6)}
\]

Our program only performs this recalculation for \( i = 0 \), since we are only interested in the power spectrum of the difference with \( \tau_0 \) — but checking these is sufficient to confirm that our theory and method are valid. Figure 6.16 shows the recomputed power spectrum of the difference for the 100 year time step.

## 6.5 A Theoretical Experiment

Our final goal is to determine the parameters of an experiment that has the ability to detect the time evolution signal in the near future. To do so, we apply the theory we developed in §5.1.2 and the implementation we discussed in §5.2.3.

We are interested in creating a “theoretical” experiment that can detect the power spectrum of the difference between the sky map now and the sky map at some near time into the future. We will set this near time to 100 years from the present for the purpose of discussion. Such an experiment would require that two identical experiments be performed — one today and another in 100 years. Recalling our discussion from §5.1.2, the parameters of a CMB experiment are the detector sensitivity \( (\sigma_{\text{det}}) \), mission length \( (t_m) \), beam profile (which we use \( \ell_s \) to parametrize), the fraction of the sky covered by our experiment \( (f_{\text{sky}}) \), which we take to be 0.75, and the number of detectors in the array \( (N_{\text{det}}) \).

It is expected that by the year 2010, detectors will be built with \( \sigma_{\text{det}} = 40 \mu K\sqrt{s} \) (Bock et al., 2006), so we will set this value for our first experiment. In addition, we assume that the number of detectors in the array must be a perfect square, since a square array of detectors is a reasonable design to actually construct. To simplify further, we will assume that \( (N_{\text{det}})^{1/2} \) is a power of two, a design choice that is often used in CCD cameras today. Figure 6.17 shows the projected results of our first theoretical experiment. The dimensions of the error boxes are defined by the extent of the \( \ell \) bin in the horizontal direction and the error for the bin, \( \sigma_{\text{bin}} \) (as defined in equation 5.12), in the vertical direction. The vertical position of the boxes is determined by treating the recalculated power spectrum of the difference for the 100 year time step (shown in figure 6.16) as the data points of the theoretical experiment. These data points are then averaged, weighted by \( \sigma_{\ell}^{-2} \), to determine the height of the center of the error box for the bin.

The projected results of another experiment with similar results but a different parameter set is shown in figure 6.18. In this experiment, we are overly optimistic about the improvement of detector sensitivity, setting \( \sigma_{\text{det}} \) to 30 \( \mu K\sqrt{s} \). The increased sensitivity allows us to reduce the number of required detectors by a factor of four, although we had to double the mission length.

Finally, figure 6.19 shows that it is possible to attain the same experimental results with a detector array 1/4 the size of that in figure 6.18 if we are willing to wait 200 years between the two

---

5A more precise method would have chosen “experimental” data points that deviate from the recalculated power spectrum by a random number chosen from a gaussian distribution with standard deviation \( (\sigma_{\ell}) \). However, the effect due to this added bit of randomness would mostly wash out due to the binning process, so we can safely ignore it for this discussion.
experiments, rather than just 100.

This exploration shows that the time evolution effect is indeed detectable, although an incredibly precise (and large) detector is required to see it. The experiments shown in figures 6.17 and 6.18 require about 67 million and 17 million detectors, respectively. While these are not unattainable numbers in principle, it is unlikely that an array of such scale will be built in the near future — especially not for the express purpose of measuring our effect. However, we have demonstrated that our time evolution effect, which is typically neglected over time scales over which humans are around to make measurements, can indeed be measured over such a time scale.

Figure 6.17: The power spectrum of the difference for the 100 year time step, and the projected results of a simulated experiment designed to measure it.

---

8An endeavor which, we should recall, requires 100 years to complete.
Figure 6.18: The power spectrum of the difference for the 100 year time step, and the projected results of a simulated experiment designed to measure it.

Figure 6.19: The power spectrum of the difference for the 200 year time step, and the projected results of a simulated experiment designed to measure it.
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